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A Special Issue

Containing

A Selection of Original Papers

By His Former Students

Commemorating

The Seventy-Fifth Birthday

of

DR. THEODORE VON KÁRMÁN

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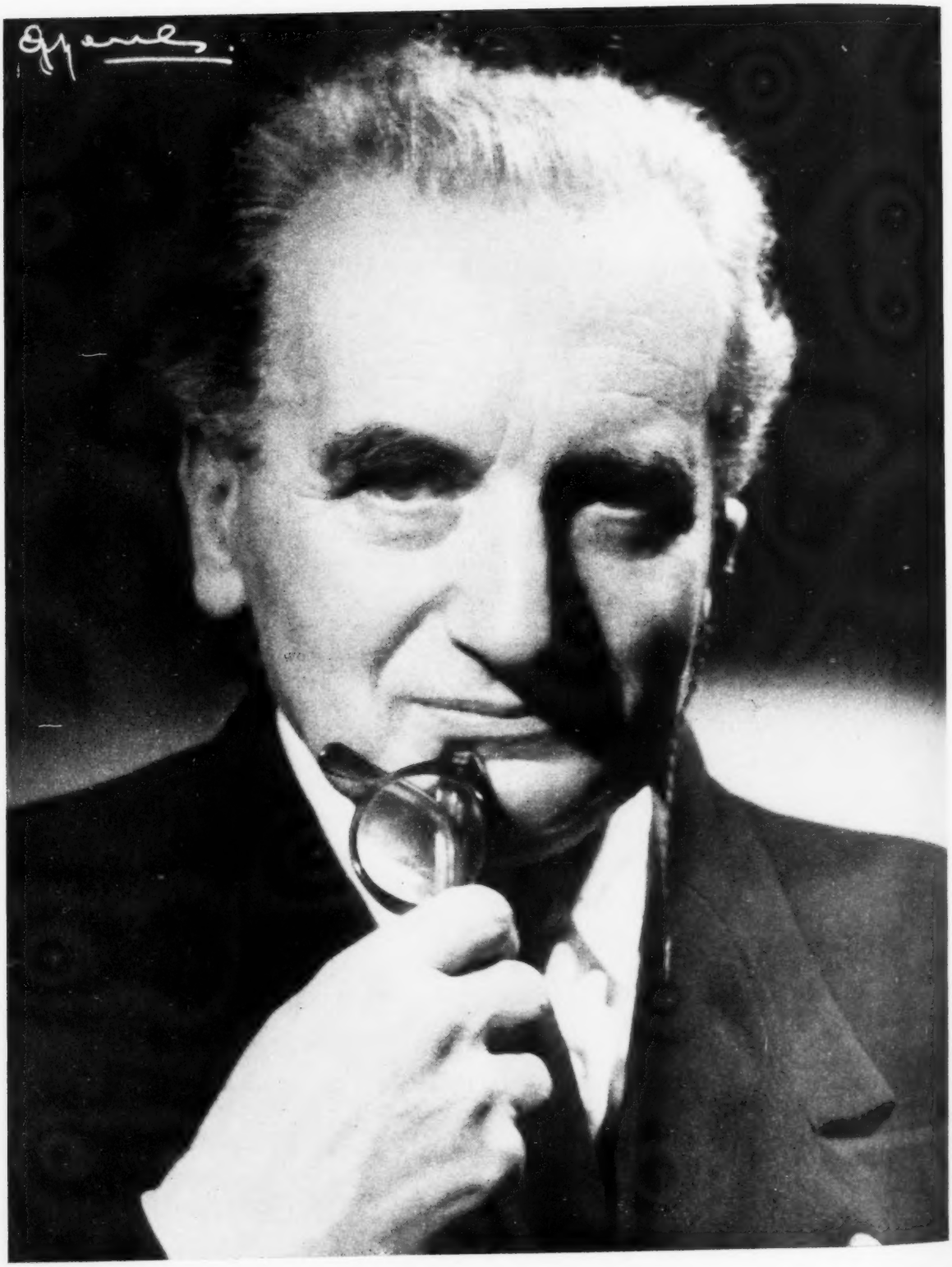

THE INSTITUTE OF THE AERONAUTICAL SCIENCES

Extends Congratulations to

DR. THEODORE VON KÁRMÁN

on the Occasion of His

SEVENTY-FIFTH BIRTHDAY

May 11, 1956

On September 24, 1926, Theodore von Kármán came to the United States for the first time at the invitation of the Daniel Guggenheim Fund for the Promotion of Aeronautics, to lecture at many universities and research institutions. A few years later, he established his residence in California, becoming a United States citizen in 1936.

No other man has had so great an impact on the development of aeronautical science in this country. Hundreds of young men became his students and scientific collaborators and were inspired to greater creative effort. They were taught by him to analyze technical problems in terms of fundamental physical concepts and to apply mathematical analysis, striving for simplicity but retaining adequate accuracy for engineering purposes. Many of these men are now leaders of aeronautical science and engineering.

Von Kármán's influence on aeronautics has by no means been confined to the United States, for he was an international personality at the time of his first visit. He has promoted international cooperation and collaborated in scientific work with men from many nations. Throughout the world one finds thousands of men who treasure his friendship and kindly counsel.

On May 11, von Kármán celebrates his seventy-fifth birthday. He is still active in creative scientific work, and in the stimulation of the reviving aeronautical sciences in the NATO countries as Chairman of the NATO Advisory Group for Aeronautical Research and Development (AGARD). The Institute of the Aeronautical Sciences decided to take special notice of this anniversary by making the May issue of the JOURNAL the Theodore von Kármán Seventy-Fifth Birthday issue. The detailed arrangements were made by the undersigned planning committee. In each of the technical areas in which Dr. von Kármán has made major contributions, one of his former students was invited to submit a paper summarizing the present status of the field. Because of scheduling problems, it was necessary to limit these invitations to Dr. von Kármán's students now residing in the United States.

On behalf of the members of the Institute, the planning committee expresses appreciation to Dr. von Kármán for his inspiring leadership and friendship for the past 30 years and wishes him good health, happiness, and prosperity for many years to come.

HUGH L. DRYDEN
WILLIAM R. SEARS
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ROBERT R. DEXTER

Theodore von Kármán

CLARK B. MILLIKAN*
California Institute of Technology

FIFTEEN YEARS AGO, on the occasion of a happy dinner celebrating von Kármán's sixtieth birthday, the writer had the privilege of preparing an "Appreciation" as an introduction to the Anniversary Volume containing scientific papers by a few of his colleagues and former pupils. In this Appreciation, reference was made to the profound influence which von Kármán had had during the preceding 30 years on the development of applied mechanics, in particular in its aeronautical aspects. Although his travels had always been wide and his interests far flung, his work up to this time had been largely done in a more or less conventional academic framework, either in Germany or in this country.

Within a few months, Pearl Harbor thrust the United States into the center of the international struggle in which much of the rest of the world was already involved. For the first few years of the war, von Kármán's headquarters remained at the Guggenheim Aeronautical Laboratory at the California Institute of Technology where, in addition to his normal academic activities, he directed special training courses for civilian engineers and Armed Forces technical officers required by the emergency. During this period, he organized and became director of the Jet Propulsion Laboratory operated by the California Institute for the Army Ordnance Department and Air Corps. In 1942, he founded and became Chief Consultant to the Aerojet Engineering Corporation, whose initial purpose was the further development and manufacture of the rocket devices—primarily JATO—originally conceived at the Jet Propulsion Laboratory. He was also frequently called upon as consultant to the armed forces and industry in connection with scientific and technical aspects of the war effort.

In 1944, General H. H. Arnold, sensing the tremendous aeronautical developments on the horizon, asked von Kármán to set up a program which would ensure that the Air Force received the best possible scientific advice and direction. The latter promptly assembled an outstanding group from the leading scientists and engineers of the country, and, thus, the same year the Scientific Advisory Group of the Army Air Force was created. With the establishment of the Department of the Air Force, this became the Air Force Scientific Advisory Board (SAB), of which von Kármán remained active chairman until 1954, when his title was changed to Chairman Emeritus.

The undertaking of this new responsibility required that von Kármán center his activities in Washington rather than Pasadena, and he was accordingly granted a leave of absence from Caltech in the fall of 1944. For the next five or six years, although his scientific productivity remained as great as ever and the number of his internationally famous lectures did not diminish, the major focus of his activities was the Air Force Scientific Advisory Group and Board. During 1945, von Kármán led teams of scientists from this group in on-the-spot studies of the German and Japanese aeronautical developments and advances during the war period. The immediate result of these studies was a small volume, *Where We Stand*, comparing the American and foreign positions in a number of critical technical areas. This was followed by a collection of some thirty volumes under the general title *Towards New Horizons*, which looked in detail into all the major scientific and technological fields which might have impacts on the future development of air power and suggested procedures by which the Air Force might stimulate and utilize them most effectively. A few years later, another SAB study, the famous Ridenour-Doolittle Report, resulted in basic changes in the Air Force organizational structure which raised the research and development function to the highest levels of command and staff in the Air Force. This in turn made possible the effective implementation of many of the earlier recommendations stimulated by von Kármán. The establishment of the Air Force Institute of Technology for the training of technical officers and of the tremendous laboratories of the Arnold Engineering Development Center at Tullahoma were also direct consequences of studies and recommendations by von Kármán and his group.

During this period von Kármán was frequently in Europe and was deeply impressed with the disruption, and in many cases devastation, which the war had caused European laboratories and scientists. He became increasingly concerned with two aspects of the problem: the frustration and near hopelessness of countless able European scientists who had no opportunity of properly exercising their abilities; and the tremendous scientific potential of the free world, which was unused in the growing struggle with the communist power behind the iron curtain. Accordingly, in 1950, he recommended that the Scientific Advisory Board sponsor a meeting of directors of the major aeronautical research institutions of NATO countries to discuss the topic, "Mobilization of Scientific Effort in Western European Countries." As a result of this meeting and much additional planning and labor, the Advisory Group for Aeronautical Research and Development (AGARD) was established as a NATO

* Director, Guggenheim Aeronautical Laboratory.

agency, with the USAF as Executive Agent. Headquarters with a permanent Secretariat were established in Paris, where the first meeting of AGARD was held in May, 1952. Von Kármán was there chosen as chairman and has since continued to occupy this position.

Under von Kármán's leadership, AGARD has had a profound and ever-growing influence on the development of aeronautical science and technology in the Western world. Panels dealing with the nonclassified aspects of wind tunnel design and testing, combustion, flight testing, and aeromedicine were initially established and have since met in technical sessions once or twice a year. Numerous General Assemblies have also been convened in various NATO countries at which scientific papers have been presented and discussed. Proceedings of these meetings, as well as scientific papers and monographs, have been published in the already famous AGARDOGRAPH series. A Documentation Committee has been active in studying, and has made important contributions to, the complex problems of the effective interchange of scientific information. One of the most significant AGARD activities has been the International Exchange Program, under which Western European students and mature scientists are afforded opportunities to visit countries other than their own either for short visits or for extended periods of study and creative research.

This magnificent and effective AGARD program is a splendid example of the extraordinary combination of qualities of its founder and only chairman: the altruism and human sympathy which led to a deep concern over the tragic status of many European scientists, the profound consciousness of the vital importance of strengthening the Western world's scientific capabilities under the existing conditions of world tension, the vision and organizational insight which created the concept of a solution for both problems simultaneously, the diplomatic and even political skill which made possible the translation of this concept into a workable and effective mechanism and organization, and the personal leadership and inspiration which have resulted in the distinguished and dedicated team which so effectively carries out the AGARD missions.

During this period of intense activity on the national and international scene, von Kármán has also continued his lifelong interest in and assistance to the world of practical engineering. He has served as consultant and adviser to numerous industrial and engineering firms in this country and abroad and has played a large role in the modernization of several European aircraft and power-plant projects. In fact, in the field of rocket motors, he has been primarily responsible for the initiation of industrial activity in several European countries.

With all of his other interests and activities, the central core of von Kármán's life has always been his creative scientific work. The 60th Anniversary Volume, mentioned above, included a list of his papers totaling, as of that date, 91 titles. It is an almost incredible fact that, during the past 15 years of intense and far-flung activities, his scientific productivity has continued undiminished, so that his publications now include more than 135 books and papers. Through countless lectures in this country and abroad, he has stimulated and inspired hundreds of young scientists and students, many of whom have collaborated with him in original researches. His publications during this period show that he has continued to illuminate and clarify fields of his earlier interests and that he has also, as in the past, unlocked the doors opening to new territories which had not been explored before. In particular, he has recently laid the groundwork for the new subject of aerothermodynamics and combustion, which had hardly been conceived of before his investigations and which is now being intensively studied by many scientists.

It is hardly surprising that the tremendous range and effectiveness of von Kármán's activities should have been widely recognized. Technical and learned societies, educational institutions, national governments, and the Roman Catholic Church have presented him with honorary memberships, fellowships, degrees, awards, and medals.

Those of his colleagues and former students who are privileged to participate in this Anniversary Issue feel themselves deeply honored in being afforded the opportunity of paying tribute to the unquestioned Dean of aeronautical scientists, who has made such incomparable contributions not only to science, learning, and education but also to industrial, national, international, and human affairs in their broadest sense. The present contributors join with von Kármán's countless admirers and friends throughout the world in hoping for many more years of his unique inspiration, wisdom, and warm fellowship.

Applied Mathematics
An Art and a Science

M. A. BIOT*

Cornell Aeronautical Laboratory and Shell Development Company

What Is Applied Mathematics?

It has been said that France is divided into 40 million Frenchmen. To raise the question of definition for applied mathematics is to invite perhaps as many answers as there are applied mathematicians or rather people who call themselves by that name. There have been numerously quoted remarks on this subject, some by individuals bearing great names in mathematics. I personally recall a lecture by the late Harry Bateman in which the applied mathematician was characterized as a "mathematician without mathematical conscience." There is no doubt that the statement reflects the existence of a deep and insoluble conflict—the contradiction between the ideal of clarity, precision, and rigor of mathematics as a science and the adventurous and imaginative urge of the mind, whether it is striving for the ultimate in its impatient search for truth and artistic expression or laboring under the fierce compulsions of the pragmatic and competitive world of technology. One could understand the feelings of the artist who undoubtedly benefits from the scientific study of colors but who would be constantly reminded of proceeding with rigorous conformity to the dictates of physics and psychology.

An excellent illustration of the conflicting mental attitudes referred to here is given by Sir Geoffrey Taylor in a recent witty essay.† His advice to those who would seek counsel from a mathematician is to use the approach one would make to a child: "Make your question as simple as possible and don't be disappointed if he finds that he cannot answer the exact question you put to him but can answer a related but rather different one."

Evidently it would be a drastic fallacy to envisage this conflict as opposing the pure and the applied sciences. Indeed it has generally been the rule that many significant discoveries in pure mathematics were made by a process of trial and error which did not exactly reflect a spirit of conformity with the conventional requirements of rigor. This is not to say, of course, that rigor lies outside the realm of aesthetics, as mathematicians well know, but simply that many times, because of human limitations, it cannot be attained except through lengthy and laborious methods.

If mathematicians were to be assembled to agree on a formal definition of the field of applied mathematics, I believe the result would bear a rather embarrassing similarity to the emasculated and noncommittal resolutions that the present generation has resigned itself to accept as the normal product of international political gatherings. As a definition would surely be presumptuous, is it not preferable to recognize that we are dealing here not with an established body of science but rather with a way of life of the scientific mind in general, best understood by a description of its activities and of its peculiar creative tools and processes? It may be truly said that mathematics today is not only the very fabric and soul of the physical sciences but also their universal language, providing the essential symbols through which the key concepts are not merely conveyed, but created, molded, and refined.

If there is a conflict, we should recognize that it is not between people but that it arises in the individual and is reflected with all the gradations and shades of a spectrum—between those whose preoccupation is mainly with the creation of a systematic and rigorous body of science and those whose interest in mathematics is purely pragmatic. Such disharmonies may be sometimes trying, but we should look upon them as a sign of inner vitality. For in this dual aspect and conflict lies the most fertile source of progress, from the early attempts at land surveying in primitive agricultural societies and their influence on the development of abstract geometry to the complexities of present-day technology. The contrast is reflected on many planes in the teaching and practice of applied mathematics, as it is only partly a science and mainly a craft and an art. This is particularly apparent in the teaching, for, as experience shows, it is usually much easier for a student to acquire the formal trappings of a theory than the ability to use it and adapt it to a particular situation.

In this respect, the achievements of von Kármán as an applied mathematician are perhaps better appraised by stressing the imponderable rather than by referring to specific contributions that are amply discussed elsewhere in this issue. He gives innumerable examples of how to tackle difficult and complex problems in widely different fields, providing the student with the opportunity of living through the same creative experience and inspiring him with an indelible vision and an enduring faith in the effectiveness of such methods in pure and applied science.

* Research Consultant.

† Taylor, G. I., *Rheology for Mathematicians*, Proceedings of the Second International Congress on Rheology; Academic Press, Inc., New York, 1954.

The Interrelation Between Pure and Applied Mathematics

We can best characterize the relationship between pure mathematics and the surrounding sciences as a continuous process of cross-fertilization. From the applied fields questions constantly arise—problems of precise definition, existence, unicity, convergence, and others that are a challenge in pure logic and belong to the classical realm of pure mathematics. In fact, it is sometimes difficult to say whether some questions of this type belong to the pure or applied category. Take, for example, the problem of unicity of solution of the Navier-Stokes equations in fluid dynamics. We may consider this as a problem for a specific equation concerned with a physical problem, but obviously the question immediately arises of the properties of equations of a similar type, devoid of any physical application and of the possibility of generalizing particular theorems to more general classes of objects. One can cite innumerable occurrences of this process. One of the most striking is the recent impressive development of theories in the functional and abstract spaces which found their origin and inspiration in the familiar eigenvalue problems of classical and quantum physics. As another example in a totally different branch of physics we could mention the boundary-layer theory, whose progress is closely associated with the name of von Kármán, and which has led to a new and interesting study of the properties of differential equations when a high-order derivative is multiplied by a vanishingly small coefficient.

In turn, such progress in the domain of pure form and concept suggests new viewpoints and novel ways in the formulation of physical problems. The tensor calculus, a primary conceptual tool in the creation of the theory of Relativity, was itself the result of abstract developments and generalizations arising from the study of geometric surfaces. Here we have a typical example of the stimulation and the molding of physical thinking by purely mathematical concepts.

In this sense, the mathematical sciences may be looked upon as providing not only an indispensable and universal language for our description and understanding of nature but also the spark for the imagination, casting the experimental facts in a new light and suggesting totally new viewpoints and relationships which would otherwise have remained unnoticed.

The Heuristic and Qualitative Functions of Mathematics

This leads us to an even more striking influence of the mathematical viewpoint on physical discovery, one that is often overlooked but is becoming more apparent every day as modern physics reaches deeper into the nature of matter and energy itself. We refer to what might be called the heuristic function of mathematics—i.e., that by which it operates as a catalytic agent for new discoveries of physical laws. There exists a common misconception embodied in the popular belief that science always proceeds in steps, from the collection of unbiased and precise empirical facts to their generalization in the form of systematic scientific laws. The prevalence of this viewpoint, due perhaps to the pervasion of our elementary and undergraduate teaching by the positivism of Auguste Comte, has had an imponderable but pernicious influence on the layman's conception of creative scientific thought and on scientific progress itself. There are numerous instances of scientific discoveries where the pattern is actually reversed, and where mathematical theories determine directly the type and scope of physical measurements and observation. One of the earliest and most striking examples is the discovery of radio waves. The existence of these waves was not suspected and could not be derived mathematically from the laws of electrodynamics as formulated by Faraday and Ampère. It took Maxwell's addition of a term known as the displacement current in the equations to make it possible to derive theoretically the existence of electromagnetic radiation, later confirmed in most striking fashion by Hertz's famous experiment. It must be emphasized that Maxwell's brilliant intuition was essentially a modification of the physical laws motivated solely by considerations of mathematical symmetry, consistency, and aesthetics. Some of the most spectacular advances in physics have been consequences of this type of thinking. Its latest triumph is the experimental confirmation of the existence of the antiproton.

Having considered what we have called the heuristic value of mathematics in the discovery of physical laws, let us turn our attention to another aspect which can be characterized as the qualitative in contrast with the quantitative. It is represented by the approximate laws and the mathematical models whose object is to give a compact and simple mathematical representation of basic phenomena. In this category we have, for example, sources and sinks, the radiation of a Hertzian dipole, etc. They may or may not be realizable physically, but they furnish a precise formulation of an intuitive concept, combining the advantage of simplicity while containing the essence of the physics. Such models are further useful as building blocks in the representation of a more actual and complex situation. We are referring here more particularly to the use of such models in mathematical physics as a powerful method for acquiring insight and understanding. We shall have occasion to come back to this later in connection with the subject of engineering sciences, where the greater complexity of the problems increases the need for such models and requires an unusual skill on the part of the applied mathematician.

One might argue that it should be simple for a mathematician and an applied scientist to join efforts in such attempts, each as a specialist in his own field; but in fact this always proves to be extremely difficult because of the high degree of compromise required between the use of sufficiently simple mathematical techniques and the inclusion of the most significant physical parameters.

Before ending these remarks, I should like to add a slight note on the critical side. It may occur, and it has happened, that physical thinking has been confused by a choice of inappropriate mathematical techniques. Such occurrences, however, are fairly infrequent and usually quickly corrected by subsequent investigators.

The Role of Numerical Analysis and Computers

Finally, let us consider the more obvious applications of mathematics, those that are primarily of a computational nature. This is a field in which, as a rule, the student trained in the techniques of pure mathematics finds his way more easily. I do not wish to enter into a detailed discussion of this vast field which has acquired a tremendous impetus since the advent of electronic calculators. I wish merely to point out two aspects of the numerical approach to scientific problems, which are not always given clear recognition.

One function of numerical analysis is to act as an instrument for empirical discovery. As an example we may cite the law of asymptotic distribution of prime numbers which was first formulated tentatively by Gauss after a study of prime number tables. The power of modern computing machines opens vast new possibilities along this line. Recent numerical work on the evolution of a complex dynamical system with nonlinear coupling terms promises to throw new light on the validity of the axiomatic foundations of statistical mechanics.

Aside from this role in fundamental science, another important function of the numerical work—and one which does not seem to have reached its full development—is to act as a substitute for elaborate physical experimentation. This is of particular importance in areas where the fundamental laws are well known and when it is desired to develop new principles applicable directly to more complex situations. This is, for instance, the case if we want to derive from the elementary laws of particle interaction some understanding of the general behavior of groups of particles such as atoms and crystals. Calculation of a sufficient number of cases may lead to the formulation of general laws much as they would arise from a systematic experimental program. There are of course drastic limitations in this approach. It does, however, possess the advantage over the purely experimental methods of enormous flexibility in the choice of parameters and variables. Furthermore it opens the way to the study of phenomena under conditions unattainable under physical testing, such as those of extremely high pressure and temperature.

As we shall have occasion to point out below, there seems to be a whole field as yet unexploited which lies at the borderline of pure science and engineering. As an example among hundreds, I might mention the calculation of electromagnetic radiation and scatter by various objects and antennas of elementary geometrical shape.

Finally, it cannot be too strongly emphasized that, no matter how powerful the numerical tools of analysis and the computing devices, they may predict the behavior of a particular system, but they never as such furnish an understanding or a grasp of the qualitative effect of each of the factors involved. They can never therefore substitute for the necessity of mathematical models and simplified theories. In fact, such simplified models not only are essential in the efficient planning of computing programs but act as the prime source of invention and imaginative design.

Mathematics and Engineering Education

It is perhaps in engineering that the most exciting challenge arises for broadening the use of mathematical methods. We can distinguish two major roles to be played by mathematics in the engineering field. One can be characterized as conceptual. With the ever-increasing complexity and the expanding domain of technology, it is essential for those engaged in research and development to possess broad knowledge and insight in connection with the problems investigated. It is in the very nature of mathematics to be synthetic and to provide compact formulation for a vast body of learning in widely different and apparently unrelated fields. We have in recent years become more familiar with what has come to be known as analog computers. However, it is not so much in its application to computing that the concept of the analog is most useful. There are innumerable phenomena obeying laws formulated by the same equations. Familiarity with one type of phenomenon immediately leads to understanding of a large class of others falling in the same analog category. One of the well-known elementary examples is the theory of the electric circuit with capacity and inductance. The differential equation for the electrical variable is identical with that of a spring-mass system, thus bringing the invisible electrical phenomenon within the immediate grasp of common experience.

What I wish to emphasize here is a point that seems to have been generally overlooked not only by the layman but also by some who are interested in the promotion of mathematical courses in the engineering schools—namely, the power of mathematics as an educational tool. It is clear that the example of the electric circuit cited above may be multiplied ad infinitum. To cite only a few, such models as the beam on elastic foundation occur repeatedly in a wide variety of problems governed by the same simple differential equation. Phenomena that involve diffusion, heat and electrical conduction, laminar viscous friction, etc., may all be explained rapidly if the student possesses the mastery of a single type of differential equation. Aeronautical engineers are well acquainted with the far-reaching analogy between the laws of electrostatics and magnetism, and the basic principles of subsonic aerodynamics. The more recent developments in the field of compressible flow have shown the usefulness of

the Lorentz transformation in airfoil theory, thereby familiarizing the student of aeronautics with one of the fundamental aspects of a remotely different subject—the theory of Relativity.

This interrelation goes even further than is usually apparent because many courses have been taught as specialized fields, and the classroom presentation has usually been encumbered by the addition of unessential features required by traditional approaches in form and concept. Enormous strides could be made in the training of engineering students if a unified treatment of science subjects could be introduced, eliminating much duplication and confusion while reaching simultaneously the dual goal of timesaving and a deeper understanding of the subject.

No doubt not all students possess the ability to grasp the abstract concepts involved or the imagination to master the techniques and a change in the direction indicated would certainly require a rather basic reorientation of our present way of thinking in engineering education, if only in the selection and encouragement of the more gifted student, and the creation of more than one level in the quality and scope of training.

We should realize that there has been a gradual breakdown of the barriers between the various branches of engineering and an enormous expansion of the knowledge in all fields. A more extensive use of the mathematical tool furnishes the only hope of coping with the educational problems involved in these two trends, and in our losing battle against increasing specialization.

From the viewpoint of the teacher, the job of presenting the subject matter in such fundamental fashion is also not an easy one. Many subjects have to be completely rethought. Formulating a problem or a physical theory so that all its essential features are properly represented in the simplest mathematical model is an art. It cannot be taught in systematic fashion but, like all craftmanship, must be acquired from example and mastered through practice. The difficulty lies not generally in the mathematics but in the proper evaluation of the various factors in a physical situation as to their essential or secondary character—a point that we have already stressed above in connection with mathematical models.

Not the least contribution of von Kármán to the advancement of the engineering sciences has been through his teaching. His lectures here and abroad have left their imprint on the thinking and outlook of generations of students. They have learned by direct example how complex phenomena could be reduced to the essential and how the mathematical language can be made to express ideas and formulate scientific problems with simplicity, elegance, and precision, by a skillful balance between sophistication and misleading crudeness. The fact that many of these former students, at present in leading and influential positions, have thus come to realize the power and practicability of mathematical methods is in itself a substantial factor in the progress of our present administration of engineering research and development.

During the years preceding World War II, an attempt was made to crystallize this approach to engineering problems and applied mathematics in the form of a textbook written by von Kármán in collaboration with this writer. As its title *Mathematical Methods in Engineering* infers, the emphasis was not on the teaching of mathematics or engineering but on the art of formulating problems mathematically. As such it constituted perhaps a pioneering attempt. However, in appraising the effectiveness of such writings, one should remember that, especially in the learning of a craft, the written word is never a substitute for the teacher.

Mathematics in Engineering Research and Development

We should also keep in mind that, while the subject of engineering mathematics bears strong resemblance to mathematical physics, there are also some differences in both the scope and the methods. It is clear, of course, that much of the fundamental knowledge in physics is of immediate use in the engineering sciences. However, there have been many areas of physics which have been left undeveloped. This has happened to problems that did not lie directly in the path of the major objective of physics—namely, the quest into the fundamental nature of matter and energy. It is especially true for such problems as those of boundary layer and turbulence, which, in addition to being on the sideline of fundamental physics and exhibiting considerable theoretical difficulty, have been investigated primarily because of their great technological interest. This also explains why applied mechanics and applied mathematics have been closely associated in their historical development.

And this leads us to another difference between engineering mathematics and mathematical physics. Many engineering systems are exceedingly complex. Furthermore the compromise between theoretical simplicity and accuracy generally lies at a different level from that of pure science because of the engineer's greater emphasis on the practicability of a theory and the margins of error usually accepted in technological problems. Therefore the degree of simplification required in the mathematical formulation of an engineering problem is often greater, with the accompanying demand of unusual specialized ability on the part of the analytical engineer.

One of the usual mathematical difficulties that occur in the problems of the engineering sciences in contrast to those in mathematical physics is nonlinearity. Only an extremely small class of nonlinear differential equations has been successfully treated by the mathematician. This is especially true in the case of partial differential equations, and much progress has been achieved in this field by the ingenuity of the individual worker in applied mechanics and the engineer. Among these von Kármán stands in a unique position. He has given an excellent review of such problems and methods,

including many original viewpoints, in the article *The Engineer Grapples with Nonlinear Problems.** It is also in connection with such problems that the art of linearization has its greatest value and makes the largest demand on the ingenuity and flair of the applied mathematician. The remarkable success of the linear airfoil theory, particularly in the nonstationary case, is an outstanding example of the power and fertility of this approach. Along the same line, one of the most ingenious discoveries is the so-called Kutta-Joukowski principle, which approximates the effect of viscosity on the flow around an airfoil by a potential flow with circulation, transforming an insuperable mathematical problem into a simple one and leading to many new and fruitful physical viewpoints.

The importance of the conceptual and theoretical approach should also be emphasized in connection with experimental programs in engineering research and development. A huge amount of experimental data has been accumulated in the past at enormous cost in time and energy. A large portion of this data, now buried in files and forgotten, has contributed little to technological advances because of the absence of coordinating theoretical concepts.

Having discussed the conceptual role played by mathematics in engineering, let us now consider the more evident one that can be characterized as that of a super slide rule. While this viewpoint is familiar and might be said to constitute the popular notion, I believe there are also some current misconceptions. It is true that electronic computers are fast becoming instruments of enormous power and flexibility. This, however, requires of the engineer an even higher standard in scientific training. A keen appreciation of the capabilities of the machines in terms of other available tools is necessary, as well as a competent evaluation of the reliability and accuracy of the input data. As experience shows, it is often possible to solve complex problems by skillful simplified analysis, statistical methods, or synthetic mathematical theorems without recourse to the expensive brute force procedure of the large computers. This type of approach has the added advantage of bringing out certain general principles that afford a better understanding of the problem and may contribute unexpected and useful results.

Everything points to an ever-increasing extension of theoretical and numerical methods versus trial-and-error testing in design and development. There are several reasons for this trend: first the availability of the new computing tools themselves, then the increasing cost and complexity of prototypes, and finally the constant advances in all fields of macroscopic physics and a better knowledge of the laws and principles applicable in the engineering sciences. This last factor has often been underestimated in discussions among engineers about the relative merits of the so-called theoretical and "practical" approaches.

* Fifteenth Josiah Willard Gibbs Lecture of the American Mathematical Society, 1940, Bulletin, Amer. Math. Soc., Vol. 46, No. 8, 1940.

In addition to the factors cited above, it may truly be said that a new dimension has been added to the analysis of engineering systems by the application of the methods of operations research and programing through which difficult design problems may be treated analytically for optimization, under conditions involving a high degree of compromise between various technological and even human factors.

A well-conceived development program should reduce the most expensive phase—the actual physical testing—to the minimum required by our inevitable ignorance of certain factors in the physical laws, the hazards of materials quality, and operational uncertainties. As a consequence, testing becomes more and more in the nature of a spot check and verification or a means of answering definite and specific questions.

The Future

We stand at the threshold of a new era, and the future presents unlimited horizons. We have referred only casually to the applications of mathematics to systems analysis, operations research, and programing and have made no mention of information theory, the theory of games, economics, or the biological and social sciences. In some of them, the role of mathematics is still in its early stage but holds tremendous promise.

It must be expected that the rapid progress and expansion of electronic calculators will considerably enhance the practical value of mathematical theories. With wider application of new inventions, such as the semiconductor elements, miniaturization, and the introduction of automatic coding, we may visualize the development of small-size analog and digital computers of enormous speed and flexibility, requiring no specialized skill for their operation. The widespread use of such powerful calculating tools will certainly promote new interests and advances in pure mathematics. Many problems that were only of academic interest in the past now become key questions related to the applications of the new computers.

In the educational field we should recognize three distinct areas—the teaching of mathematics to science and engineering students, mathematics as a cultural subject, and the teaching of pure mathematics and its techniques as a distinct science and specialization. We have already discussed the advantages of a more unified outlook in the training of science and engineering students. There is no doubt that considerable difficulties shall be encountered in raising the present standards, but should we not pay more attention to the selection of students particularly gifted to benefit from such a change? We must keep in mind that an important purpose of advanced teaching, as opposed to mass education, is to stimulate existing talent and help it discover itself.

In the field of research and development, progress will depend not only on the technical personnel but also to a great extent on those at the policy-making and administrative levels. Here again improved educa-

(*Continued on page 489*)

The Behavior of Nonlinear Systems

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SUMMARY

Many of the phenomena that occur in the world around us are governed by nonlinear relationships. In the development of the mathematical sciences, the difficulties of nonlinear analysis have hindered the formulation of nonlinear concepts that would permit us to understand such phenomena. In the present article, our progress in understanding the behavior of nonlinear systems is reviewed and an attempt is made to present the resulting concepts in such a way that they may be applied with some generality to other problems.

INTRODUCTION

IF ONE LOOKS BACK over the long history of accomplishment in the sciences, he is struck by the fact that most of our effort has been centered on linear systems and linear concepts. If, at the same time, one takes a look at the world around us, he is everywhere confronted by phenomena that are nonlinear in their behavior. Linear concepts give only a superficial understanding of much that we see in nature. If we are to make our analyses more realistic, we must acquire greater facility in comprehending and using nonlinear concepts.

In recent years, we have developed large computing machines, and, in many instances, we have fed nonlinear problems into these machines in the hope that the resulting solutions would give us a better understanding of nonlinear behavior. In general, we have found that simply grinding out solutions gives us little more understanding of nonlinear behavior than, for example, watching nature herself grind out solutions for such nonlinear problems as the weather. Understanding lies not in terms of equations or solutions but rather in terms of fundamental and well-understood concepts. For example, two engineers sit in an airplane during takeoff. The ash tray starts to buzz. One looks at the other and says "Resonance!" The other nods and both believe they understand the phenomenon. We understand nature only when we can piece together its behavior in terms of concepts so simple they may be well understood and so broad they may be stated abstractly without reference to a particular situation. The list of such concepts is long and includes such terms as resonance, hysteresis, waves, feedback, boundary layers, turbulence, shock waves, buckling, weather-fronts, contagion, immunity, inflation, and depression. Many of our most useful concepts are nonlinear in character, and a part of our inability to express in the exact language of mathematics such everyday phenomena as the flow of water in the gutter or the curling of smoke from a cigarette lies in the fact that we have been reluctant to plunge in and understand nonlinear mathematics.

We search for simplicity in nature but we are not always sure what we mean by simplicity. We find that there are certain nonlinear concepts which at first appear complex, but which, when understood, are found to have universal applicability. In such cases, it is more profitable to master the concept in its nonlinear aspect than to patch together strained linear concepts on the ground that linear concepts are inherently simpler. It appears to be imperative to increase our store of nonlinear concepts and to put them on a firmer mathematical foundation. It is the purpose of the present article to review a few of the nonlinear concepts that we have succeeded in establishing and to present them in a way that they may be used with reasonable generality.

What is it we expect of mathematics in the sciences? All around us we see action and interaction that is subject to quantitative statement. Since we can not hope to analyze all we see in one grand analysis, we are forced to view the universe as a collection of systems which we can analyze individually. When we carve out such a system, we sever the interactions with the rest of the world, and these must be replaced as inputs and outputs of our system. Each system can be viewed in the light of the analogy suggested by Maxwell of a machine in a closed room, the only connection to the exterior of the room being by means of a number of ropes which hang through the floor into the room below. Our desire is to determine the behavior of the machine overhead, but this may be done only by manipulating the various ropes which hang through the ceiling. The question is whether by pulling on the ropes and by correlating their response in an intelligent way, the complete behavior of the rope system may be predicted when an outsider comes in and pulls on the ropes in any manner he pleases. Modern electronics people have replaced Maxwell's room by a "black box" and have taken the additional step of designating some of the connections as inputs and others as outputs during a given operation.

In well-established areas of science, if we look inside the room or black box, we can recognize the elements we see inside; and since we know their individual behaviors, we can frequently synthesize these behaviors to give the behavior of the entire collection. However, on the forefronts of knowledge, we do not recognize what we see when we look inside a system such as an atomic nucleus or a virus or turbulence or the human brain, and we are forced to join theory and experiment into the operational approach described by Maxwell.

* Chairman, Department of Aeronautics.

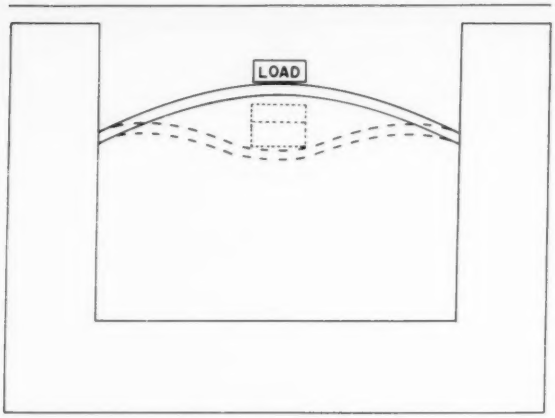


FIG. 1. Loaded flat arch.

CLASSIFICATION OF SYSTEMS

Linear and Nonlinear Systems

What do we mean by a linear system? Linear of course implies a straight line relationship between cause and effect—i.e., between input and output or between independent and dependent variables. A system is linear if all the effects are in proportion to the causes. In a looser terminology we say that an effect is proportional to a cause if both simply increase or decrease together. Justification for such usage lies in the fact that by distorting the scale for measuring either variable, the relationship can be made linear. Nonuniform marks placed on a stick can indicate directly the volume of liquid in a spherical tank. However, departure from linearity cannot always be handled so easily. The flat arch shown in Fig. 1 deflects proportionately to the load as long as the load is small. However, at a critical load, the arch will "pop through" from A to B on the force-deflection curve of Fig. 2. Upon removing the load, the return jump does not occur from B to A but from D to C. Such a system is "essentially" nonlinear in that no change of scales can eliminate the jump or the hysteresis. While such jumps and hysteresis cannot occur in linear systems, we shall see that they enter quite naturally in nonlinear phenomena.

Kinematic and Dynamic Systems

If a system contains only such elements as gears, pulleys, resistances, rectifiers, relays, etc., the relationship between inputs and outputs is of a kinematic nature which can be stated in algebraic terms without derivatives or integrals. Such a relationship can be presented as a surface in a many dimensional space whose coordinates are the inputs and outputs. If we add masses, springs, inductances, condensers, and the like, we have a dynamical system which needs differential equations rather than algebraic equations to express the relationship between inputs and outputs. If we move the inputs and then hold them fixed, the outputs will continue to move. The system has acquired a memory and its present behavior is influenced by its past history. Its behavior can no longer be represented by a surface in a many dimensional space but must be expressed in terms of integrals which sum up all past influences on the system.

Along with this change in the behavior of the system we must alter our concept of independent and dependent variables. The flat arch is a kinematic system in which the load is the input or independent variable and the deflection is the output or dependent variable. Now consider a system represented by

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = V$$

where the voltage, $V(t)$ is the input forcing function and the current $i(t)$ is the output response. For a kinematic system, we should call V the independent variable and i the dependent variable. But what role does t play? It is an independent variable, but quite different from V. We resolve this situation by calling t *the* independent variable, and we allow V to achieve independence by acquiring the right to be *any* function of t. That is, we now seek to determine the response, $i(t)$ for all possible programs for the input, $V(t)$.

Linearity now takes on a more subtle meaning. The electrical system given above is still linear even if the coefficients L, R, and C vary with the time. Note that they must vary according to an explicit program determined beforehand. If L, R, and C are influenced by the action taking place—i.e., by i or V—the system is nonlinear.

The Properties of a System

The human mind so readily grasps the notion of linearity that we endow the coefficients of proportionality of a linear system with special significance and

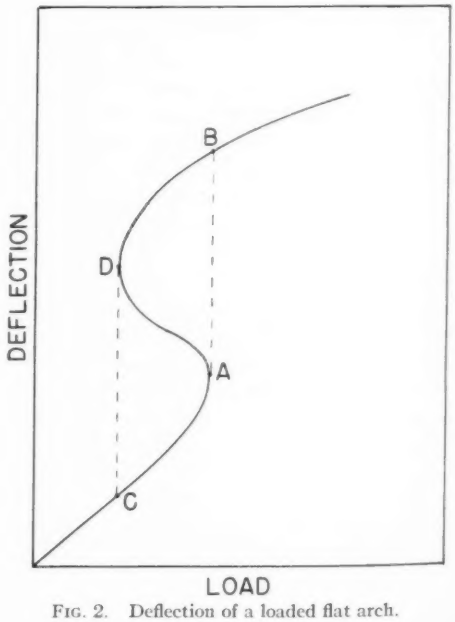


FIG. 2. Deflection of a loaded flat arch.

call them the properties of the system. Thus, the stiffness of a spring is the constant of proportionality between force and deflection, resistance is the constant between voltage and current, mass between force and acceleration, viscosity between stress and rate of strain, conductivity between heat flow and temperature gradient, and so on.

We have just seen that a system may be linear even though its coefficients of proportionality—i.e., its properties—change with time. This concept of the properties of a system is so important that we sometimes cling to it even when the system is nonlinear and the coefficients vary with the behavioral variables. Thus in a saturable inductor we say that the inductance depends on the current, or in a discharge tube we say the resistance depends on the current. Strictly speaking, it is nonsense to say that we may write a function $a(x)$ as $xb(x)$ and then call $b(x)$ a property of the system. However, it is worthwhile to retain the intuitive advantages of a linear coefficient when it can be put on a firm foundation by such stratagems as local linearization when the changes are small or slow or by an overall or equivalent linearization when the changes are large or fast.

Independent Variables

Other independent variables than time occur in physical problems. If an elastic column with bending stiffness B is loaded with axial forces P at each end, the differential equation for the deflection y is

$$\frac{d^2y}{dx^2} + \frac{Py}{B}\sqrt{1 - \left(\frac{dy}{dx}\right)^2} = 0$$

Here the independent variable is x, the distance along the column. The input to the system is P and the output is y. Notice that even if we neglect $(dy/dx)^2$ compared to unity and write the above equation as $d^2y/dx^2 + Py/B = 0$, so that we may treat it by linear methods, it is still a nonlinear equation between input P and output y. As the load is increased, no deflection takes place until a critical value is reached, and then the column buckles. The resulting nonlinear relation between the input P and the abridged output y_{max} is shown in Fig. 3 for both the approximate and the exact equations. The nonlinearity of the approximate equation manifests itself as an eigenvalue problem which singles out a discrete set of values of P which satisfy the end conditions.

When physical systems have several independent variables, we must turn from ordinary to partial differential equations. At first glance, one would suppose that the lore built up for systems of one independent variable would be readily carried over to systems with several independent variables. Historically such does not appear to be the case with nonlinear phenomena. Work on nonlinear systems with a single variable (usually time) developed largely in the field of analytical mechanics and electrical engineering. Work on nonlinear systems with several independent variables

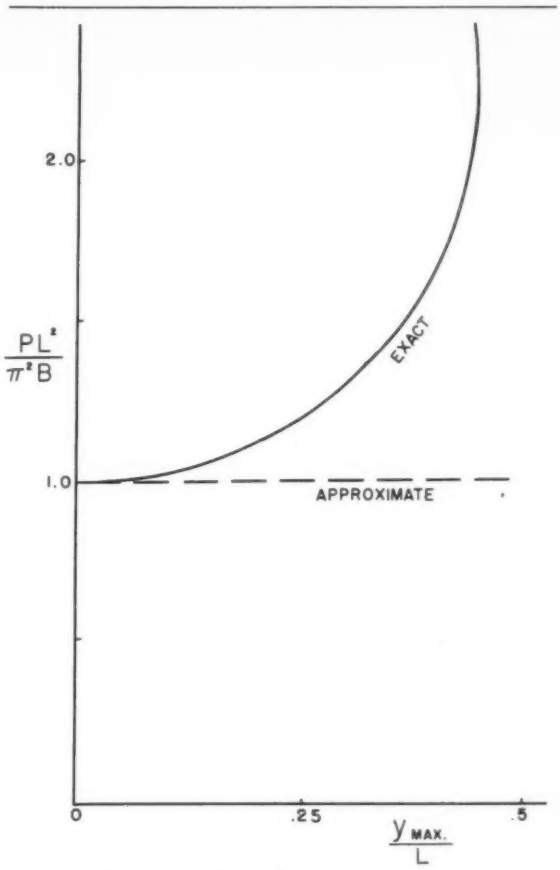


FIG. 3. Buckling of an elastic column.

developed primarily in the fields of solid and fluid mechanics. Few concepts from either of these separate efforts have been carried over to the other. Moderate progress has been made in abstracting and generalizing the concepts developed for ordinary differential equations so that they may be used with some universality. However, because of the greater difficulties involved, most nonlinear concepts of partial differential equations still are in encyclopedic form. Their generalization presents a challenge to the physical scientist.

It is sometimes convenient to use more sophisticated quantities than space or time as independent variables. In fluid mechanics, a hodograph space in which the velocities are independent variables is frequently used. The list of possible independent variables also includes such quantities as energy, momentum, frequency, wave number, and a large collection of dimensionless variables. For simplicity in what follows, we shall deal with independent variables as though they are in space and time even though the class of such variables is much larger.

Autonomous and Nonautonomous Systems

If the properties of a system are not affected by the dependent variables, we say that the system is linear. On the other hand, if the properties of a system are not

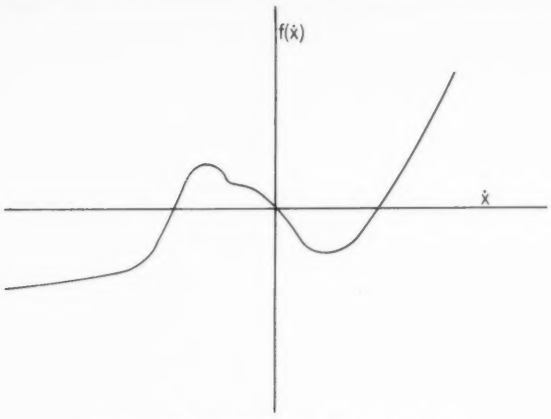


FIG. 4. Typical nonlinear damping function.

affected by the independent variables, we say the system is autonomous. This may be described physically as saying that if the action of the inputs or outputs does not change the properties of a system, it is linear; and if the system possesses no clock* that changes the properties according to a fixed program, the system is autonomous. For example, an ordinary pendulum swinging through small angles is a linear device. When the amplitude is large it becomes a nonlinear device. In both cases it is autonomous. If the length of the pendulum is shortened according to a fixed program of the time, it is nonautonomous. As such it can either be linear or nonlinear.

In general, it may be said that if a system is autonomous, its behavior is uninfluenced by when or where it takes place in time or space. This is because the independent variables enter only as differentials, and consequently they may be shifted by a constant amount without any effect. It will at once be recognized from this criterion that many familiar systems are autonomous.

Behavior of Systems with a Single Independent Variable; Ordinary Differential Equations

Behavior of Isolated Autonomous Systems

One of the simplest questions that can be asked concerning the behavior of physical systems is whether it is possible for a system to exhibit an interesting behavior if it is isolated† in such a way that it receives no input stimuli from outside. It is likely that everyone will not agree on what constitutes interesting behavior of such a system. A reasonable definition would appear to be that any spontaneous or self-sustaining action would be interesting, whereas a lack of action would be uninteresting.

First let us see what linear systems have to offer. Regardless of how many variables a linear autonomous system has or how high-order derivatives are involved, its behavior must fall into one of three patterns. Its action must die out, or grow without limit, or oscillate with a combination of sine waves. The last two possibilities must be ruled out for physical systems. No system can exhibit an action that grows without limit. Eventually it must strike stops, burn out a fuse, or otherwise become nonlinear. For a linear system to oscillate without damping, it must have certain grouping of its coefficients precisely equal to zero, which is impossible in real systems. Consequently, all autonomous linear systems show no interesting behavior by themselves.

When we introduce nonlinearity, the picture is changed completely. Consider a system whose equation is $m\ddot{x} + f(\dot{x}) + kx = 0$ where $f(\dot{x})$ is a nonlinear function of the type shown in Fig. 4 having a slope which is negative through the origin and is positive for numerically large values of $\dot{x}$. For small amplitude oscillations, the damping is negative and the oscillations will grow. On the other hand, large oscillations will be diminished by positive damping. The motion will be trapped somewhere in between, and the system will necessarily perform a self-sustained action, which puts it in the interesting class.

To investigate the behavior of such systems in more detail, let us suppose that $f(\dot{x})$ has the polygonal shape shown in Fig. 5. Nonlinear solutions may be obtained by patching together known linear solutions. However, a simple bookkeeping difficulty develops in trying to understand the general behavior of the system. If we plot our results as x vs. t, the graph becomes covered with a maze of patched curves for different initial conditions and it is difficult to deduce any overall behavior pattern. However, since the system is autonomous, t may always be replaced by $t - c$. Thus we know how one constant of integration occurs, and we are able to use this knowledge to reduce the order of the equation and hence the complexity of the initial conditions. We write our original equation as $m\dot{y} = -kx - f(y)$ with $\dot{x} = y$. We view these as two simultaneous equations for x and y. If we divide one

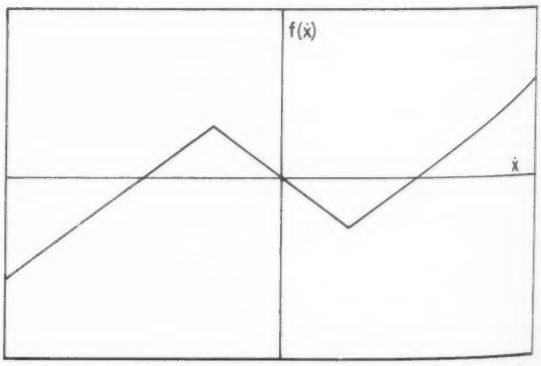


FIG. 5. Nonlinear damping function composed of straight line segments.

* Here, the word "clock" is used to denote any generic device for establishing functions of the independent variables whether they be time, distance, or other quantities.

† Here we mean isolated in an information-theory sense of the word rather than in an energy sense of the word.

by the other, we eliminate t giving $dy/dx = -[kx + f(y)]/my$ which is an equation of lower order. If we plot our results in an x, y phase plane, a single family of curves gives the behavior pattern of the system for any combination of initial values of position x and velocity y.

If the damping were zero, we could integrate the above equation directly to give the energy equation $my^2/2 + kx^2/2 = E$ with E being the constant of integration. In the phase plane, with proper scales, this solution is a series of circles for different E as shown in Fig. 6. If this solution is substituted back into the equations above, we obtain the familiar result

$$x = A \sin (\sqrt{k/m}\, t + \phi)$$

$$y = A\sqrt{k/m} \cos (\sqrt{k/m}\, t + \phi)$$

If linear damping is added, the trajectories spiral in to the origin.

For the polygonal damping function of Fig. 5, we may piece together spirals corresponding to the three linear segments of $f(y)$. In each of the three regions, the appropriate trajectories are drawn and joined at the boundaries as shown in Fig. 7. In the central band, the trajectories spiral outward while in the upper and lower regions they spiral inward. At an intermediate amplitude, there is a closed curve which spirals as much outward as inward. All other trajectories approach this limit cycle which represents a stable oscillation performed by the system of its own accord.

The above type of behavior is not confined to the specific damping function of Fig. 5 but exists for a great variety of nonlinear functions as we shall see presently. As we increase the complexity of the nonlinearity, we obtain ever more fascinating patterns of behavior. One of the first to explore the behavior of such nonlinear systems was Poincaré whose work[1, 2] was so far-reaching that it has formed a base for almost all that has followed.

In linear systems, the concept of stability has meaning only at equilibrium points. In the above nonlinear example, the system performs a stable motion even though it is not in equilibrium. It is not easy to formulate this new concept with mathematical exactitude and physical generality. Those interested in this aspect should consult the pioneering work of Liapounoff.[3]

In order to explore more complex ways in which nonlinearity can effect the behavior of systems, consider the following graphical method which is an extension of that given by Liénard.[4] It covers a wide range of interesting cases. We start with the two equations $dx/dt = y - \phi(x)$, $dy/dt = -x + \psi(y)$. Nonlinearity enters through the two functions $\phi(x)$ and $\psi(y)$ whose only restriction is that we be able to draw them. If coefficients occur on the linear terms, they may be eliminated by rescaling x, y, and t.

In the x, y plane, we plot $\phi(x)$ and $\psi(y)$ using the y axis for ϕ and the x axis for ψ. In Fig. 8, we move horizontally from a typical point x, y to the ψ curve

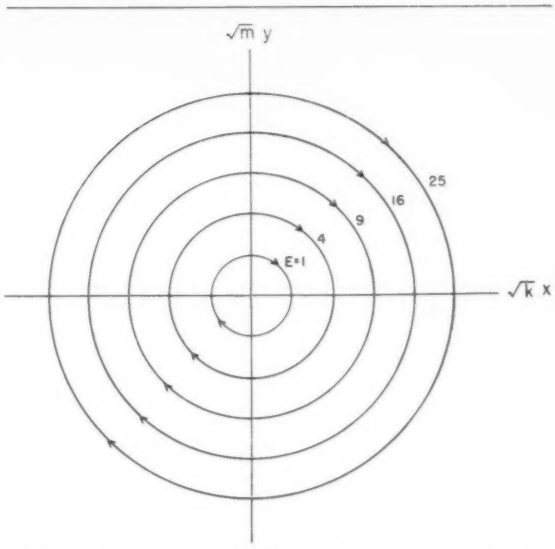


FIG. 6. Phase plane trajectories of an undamped oscillator.

and then down to meet a similar line which drops down from x, y to ϕ and then goes horizontally to the intersection. A short arc through x, y with center at this intersection has the slope, $dy/dx = [\psi(y) - x]/[y - \phi(x)]$ as required by the above equations* with t eliminated. In this way, trajectories may be quickly drawn for any ϕ and ψ.

Now let us consider the case of the sliding friction between a block and the surface on which it is resting. If the block is stationary, the frictional force can be any value between the plus and minus limits of static friction. Once the block begins to move, the dynamic friction drops to a value somewhat less than the static friction and is nearly constant. Actually, it usually drops a little for moderate speeds and then eventually rises for larger speeds as shown in somewhat exaggerated form in Fig. 9.

If the block is restrained by a spring, its oscillations will be so highly damped as to be uninteresting. However, if the surface on which it is placed is moving steadily, its motion can be quite interesting. It is relatively easy to put the equations for the resulting motion in a form so that trajectories may be drawn by means of the graphical method just given. Such trajectories are shown in Fig. 10 in which the x and y axes have been interchanged so that the frictional curve may be readily compared with Fig. 9. The steady velocity displaces the curve by an amount V and shifts the equilibrium position to the point A. Because of the negative slope of the friction curve, this equilibrium point is unstable and any small motion will spiral outward until the velocity becomes great enough to bring the block to rest on the moving surface. The block is then carried to the point B where the spring force overcomes the static friction and the block goes through

* The above construction is readily generalized for different combinations of $\pm x$, $\pm y$ on the right-hand side of the above equation.

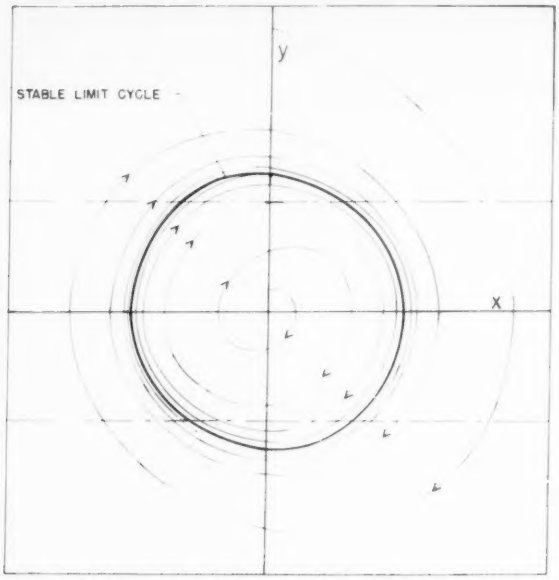


FIG. 7. Phase plane trajectories of a nonlinear oscillator showing limit cycle.

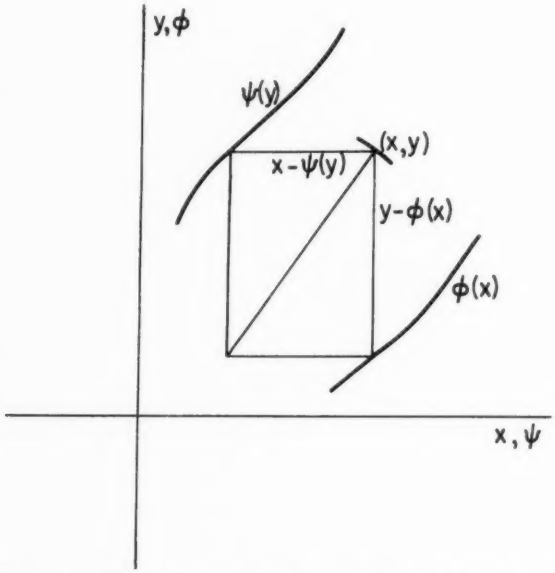


FIG. 8. Method of graphically constructing trajectories of nonlinear equations.

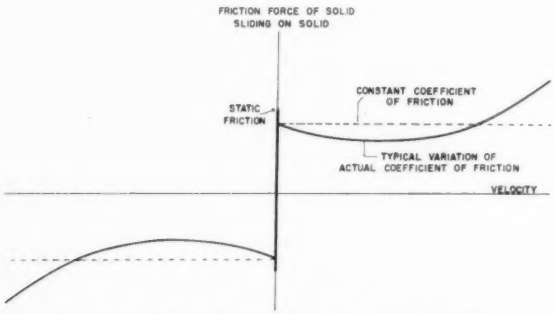

FRICTION FORCE OF SOLID SLIDING ON SOLID
STATIC FRICTION
CONSTANT COEFFICIENT OF FRICTION
TYPICAL VARIATION OF ACTUAL COEFFICIENT OF FRICTION
VELOCITY

FIG. 9. Effect of velocity on solid sliding function.

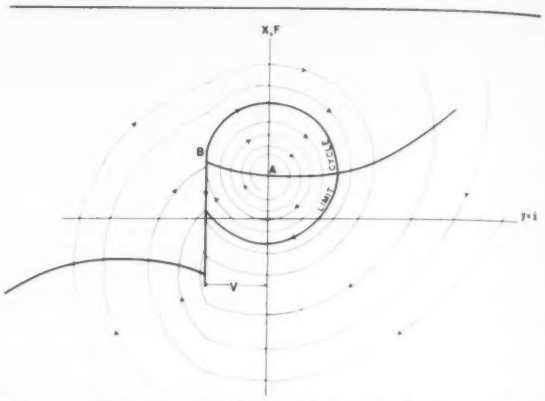


FIG. 10. Phase plane trajectories showing oscillation caused by sliding function.

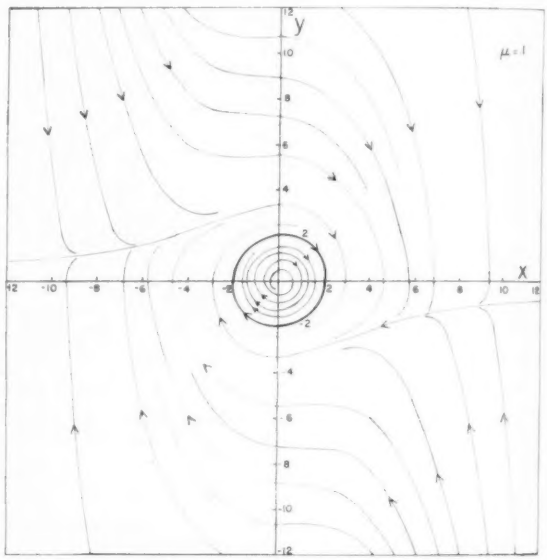


FIG. 11. Phase plane for Van der Pol oscillator with $\mu = 0.1$.

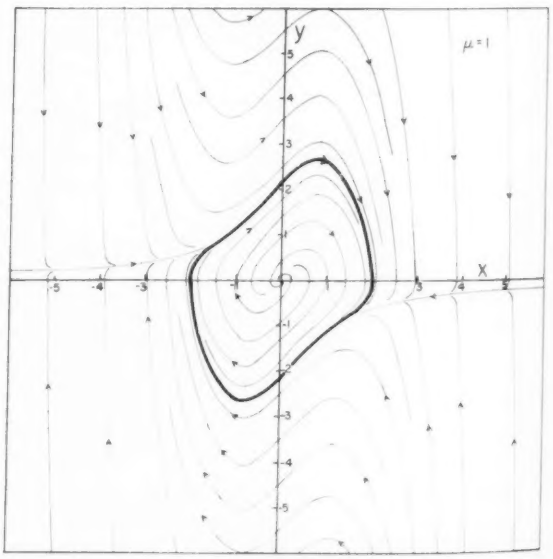


FIG. 12. Phase plane for Van der Pol oscillator with $\mu = 1$.

a cycle of oscillation until it is again trapped on the moving surface and the process repeats itself. If the block is given a large initial displacement or velocity, it quickly slows down so that it oscillates on the limit cycle. The nonlinearity of the sliding friction thus serves to drive the block in a sustained stable oscillation, a latent possibility that would not have been anticipated from its behavior on a stationary surface.

We now see why a violin string is forced to vibrate when it is stroked with a bow, why doors creak, brakes chatter and groan, how crickets chirp, and why a fingernail, rubbed on a blackboard, gives an unpleasant sensation.

The behavior of many of the common oscillators around us such as clocks, buzzers, blinkers, radio oscillators, whistles, musical instruments, and the like can be explained by nonlinear analyses similar to the one just given. All these show that nonlinear systems can exhibit interesting behaviors even when they are isolated.

One of the simplest analytic expressions for nonlinear damping is that used in the much studied Van der Pol equation $\ddot{x} - \mu\dot{x}(1 - x^2) + x = 0$. This equation has the property that when x is small, the damping is negative, but becomes positive for large x. Trajectories of the Van der Pol equation may be mapped out by our graphical procedure using $\psi(y) = 0$, $\phi(x) = \mu x(1 - x^2/3)$ or by the method of isoclines used by Van der Pol.[5] In Figs. 11, 12, and 13, are given the phase trajectories for $\mu = 0.1$, 1, and 10

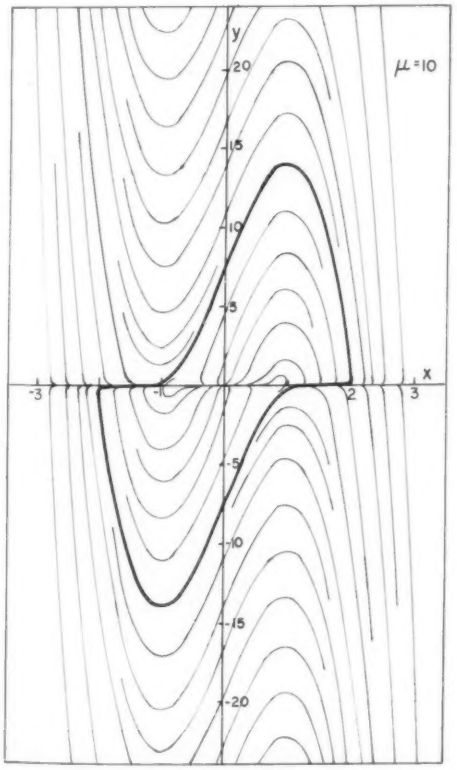


FIG. 13. Phase plane for Van der Pol oscillator with $\mu = 10$.

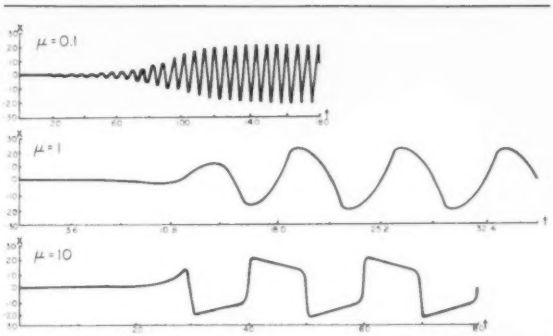


FIG. 14. Displacement-time curves for Van der Pol oscillators.

corresponding to small, moderate, and large influences of the damping function. In Fig. 14 is given the corresponding plots of amplitude versus time for motion growing into a limit cycle from a small disturbance.

In the Van der Pol equation, the nonlinear term is $\mu\dot{x}x^2$. It is of interest to consider the effect of more complex nonlinear terms involving several derivatives of various orders. If the effect of the nonlinearity is not large, we may follow the method of equivalent linearization of Kryloff and Bogoliuboff, and write the equation of motion as $m\ddot{x} + \mu f(x, \dot{x}, \ddot{x}) + kx = 0$ where μ is sufficiently small that the motion in phase-plane coordinates, $\sqrt{m}\,\dot{x}$ and $\sqrt{k}\,x$, is nearly circular. After one cycle of the motion, the effect of the nonlinear function, f, will prevent the phase point from returning to its starting point on the original circle. We may interpret this change in radius—i.e., amplitude—as caused by an equivalent linear damping and the change in angle—i.e., delay—as caused by a change in the mass or spring constant. By integrating around a cycle, it can be shown that the equivalent linear damping is

$$\lambda = (\mu/\pi A)\int_0^{2\pi/\omega} \cos\omega t\, f(A\sin\omega t, A\omega\cos\omega t, -A\omega^2\sin\omega t)\,dt$$

and the change in mass or spring constant is

$$\Delta k = \omega^2\Delta m = (\mu\omega/\pi A)\int_0^{2\pi/\omega} \sin\omega t\, f(A\sin\omega t, A\omega\cos\omega t, -A\omega^2\sin\omega t)dt$$

Here A is the amplitude of the motion and $\omega = \sqrt{k/m}$ is the circular frequency. If we adopt electrical engineering terminology, the nonlinear function has an equivalent complex impedance given by

$$Z = (\mu/\pi A)\int_0^{2\pi/\omega} e^{-i\omega t} f(A\sin\omega t, A\omega\cos\omega t, -A\omega^2\sin\omega t)\,dt$$

As examples, the Rayleigh damping function, $-\dot{x}(1 - \dot{x}^2/3)$, has an equivalent resistance of $-(1 - A^2\omega^2/4)$, and the Van der Pol function, $-\dot{x}(1 - x^2)$, has an equivalent resistance of $-(1 - A^2/4)$. By equating these resistances to zero, we obtain the amplitude of the corresponding stable limit cycles. In linear

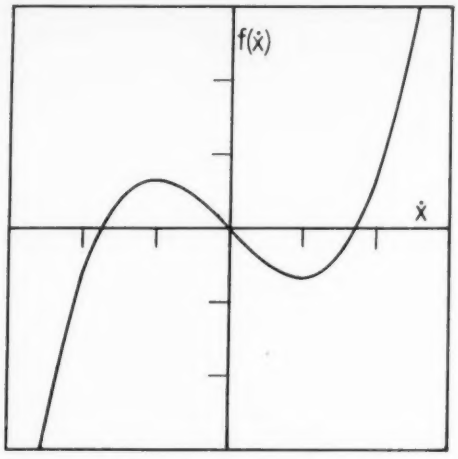


FIG. 15. Nonlinear damping function $-\dot{x}(1 - x^2)$.

systems, the impedance varies only with the frequency whereas in nonlinear systems it depends on both frequency and amplitude. In the above analysis, we are ascribing properties to nonlinear systems similar to those we use for linear systems. If used with care, such nonlinear properties can be treated in the same analytical and intuitive ways as linear properties except it must be remembered they are amplitude dependent.

Consider a nonlinear term of the form $(d^\alpha x/dt^\alpha) \cdot (d^\beta x/dt^\beta) \cdot (d^\gamma x/dt^\gamma)$ where negative α, β, or γ is to be interpreted as integration. It is not difficult to generalize the above analysis to include such a term, and the equivalent impedance is found to be $\sigma A^2(i\omega)^{\alpha+\beta+\gamma-1}$ where σ is a numerical factor depending on α, β, and γ. Such a term acts as a resistance or a reactance according to whether $\alpha + \beta + \gamma$ is odd or even. As far as amplitude characteristics are concerned, such a term is equivalent to x^3 or $\dot{x}^3$ depending on $\alpha + \beta + \gamma$. Similarly, a product of any n derivatives is equivalent to x^n or $\dot{x}^n$ as far as amplitude characteristics are concerned. We readily build up nonlinear functions by forming sums of products of the various derivatives. Let us consider the behavior of systems having as damping terms the following sequence of functions:

$$-\dot{x}(1 - a\dot{x}^2); \dot{x}(1 - a\dot{x}^2 + b\dot{x}^4); \dot{x}(1 - a\dot{x}^2 + b\dot{x}^4 - c\dot{x}^6)$$

These functions have been plotted in Figs. 15, 16, and 17. We have seen that a function such as the one in Fig. 15 will lead to a stable limit cycle similar to that of Fig. 7. The function in Fig. 16 will yield two limit cycles, one stable and the other unstable as shown in Fig. 18. This system will act quite lifeless near the origin, but if disturbed violently, it will break into a stable oscillation that might otherwise go completely unnoticed. Such "hard" oscillators requiring a threshold disturbance to set them off are not uncommon in nature.

The function of Fig. 17 provides negative damping near the origin and a succession of regions of positive and negative damping such that the system will have two stable limit cycles separated by one unstable limit cycle as shown in Fig. 19. This system will exhibit "soft" oscillations which start from the origin and approach the inner limit cycle. However, if the system is violently disturbed, it will jump over the barrier of the unstable limit cycle to the large stable limit cycle and oscillate in a quite different way.

In a similar manner, we may determine the effect of sums of terms involving products of derivatives of any order. Furthermore, we may obtain an indication of system behavior even when the functions are not analytic but are merely topologically similar to analytic functions with regard to the location of their zeros and maxima.

Before we finish our discussion of isolated autonomous systems, it is of interest to discuss the phenomenon called "relaxation." We use as an example the neon tube circuit of Fig. 20. If A is the ionizing voltage of the tube, then for any initial lesser voltage, no current will flow through the tube. However, if the voltage A is exceeded, the steady-state current is as shown in Fig. 21.

If the switch of the circuit is closed, current will flow through the resistor, charging the condenser. Initially no current flows through the tube, but if the battery voltage exceeds the ionizing voltage, the tube eventually will start to conduct. At this point it might be expected that the voltage-current relationship of the tube would follow along the solid curve of Fig. 21. However, here we have a situation similar to the flat arch discussed earlier. The voltage across the tube must continue to rise because the battery is still charging the condenser, but the voltage can not rise if it follows the

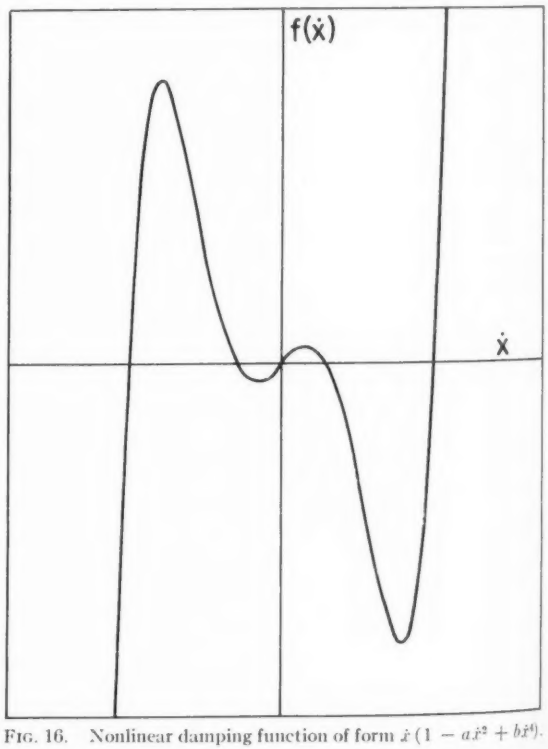


FIG. 16. Nonlinear damping function of form $\dot{x}(1 - a\dot{x}^2 + b\dot{x}^4)$.

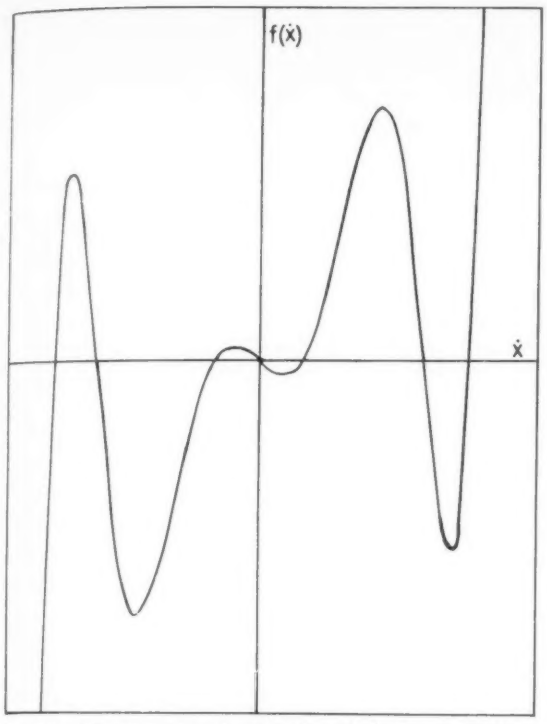


FIG. 17. Nonlinear damping function of the form $-\dot{x}\ (1 - a\dot{x}^2 + b\dot{x}^4 - c\dot{x}^6)$.

solid curve. The system abruptly jumps to the condition at B, analogously to the buckling of the flat arch.

When current is flowing through the resistor, it reduces the voltage from the battery along the dashed line EF. At B, the tube must discharge the condenser and the voltage falls to the point C. Here we have a second dilemma. The voltage must continue to fall, but it can not do so and stay on the solid curve. At this point, the flow discharge ceases, the system jumps to D, and the cycle starts over again. In this way a passive nonlinear element can lead to a self-sustained oscillation involving jumps and hysteresis.

During the jumps from A to B and C to D, we neglected any inductance in the circuit containing the tube. Such inductance may be small but it is not zero. If we call this inductance L and designate the steady nonlinear voltage-current relationship of the tube by $U(i)$, the equations for the system are $dV/dt = (E - V - iR)/RC$ and $di/dt = [V - U(i)]/L$. If we use V and i as phase plane variables and rescale them to absorb the constants, we can use the graphical method given earlier to trace out the trajectories. Because of the smallness of L, the rescaling will greatly compress the scale for i, and after the trajectories are obtained, this scale must be reëxpanded to obtain an undistorted picture of the system's behavior. This has been done for a typical case in Fig. 22. The trajectories are nearly horizontal everywhere except near the curve $V = U(i)$. Here they bend rapidly to channel into the limit cycle previously discussed.

If we eliminate t from the two equations, we have

$$\frac{dV}{di} = \frac{LC}{R}\left[\frac{E - iR - V}{V - U(i)}\right]$$

If L is extremely small, then dV/di is nearly zero except very near the curve $V = U(i)$. Consequently the equation has the simple solution $V =$ constant everywhere except within a thin boundary layer on both sides of the curve $V = U(i)$ where the complete equation must be used.

If in the above equation, L is made vanishingly small and the possibility of large di/dV is excluded, the equation reduces to $V = U(i)$, implying that the system must follow the characteristic curve of the neon tube. However, in passing to the limit we have lost the possibility of assigning a constant of integration, since we now have an algebraic equation rather than a differential equation. When the system jumps or when it is started off the characteristic curve, the differential equation must be used to obtain a proper picture of the behavior. A similar situation exists for the flat arch described earlier. During the buckling of the arch, the load-deflection relation changes from a kinematic equation to a dynamic equation.

How can we tell ahead of time that a system will experience discontinuous jumps in its behavior and that some essential terms of higher order are being neglected? The symptoms of such behavior are as follows. We have a relationship between the variables of the problem such as shown in Figs. 2 or 20 where a maximum or minimum exists for one of the variables. In addition, we have the requirement* that this variable, in passing through its maximum, must continue to increase (or decrease for a minimum). Since such a requirement is an apparent impossibility and yet since the requirement is real, the answer invariably

* Imposed from outside in the case of the arch, self-imposed in the case of the neon tube.

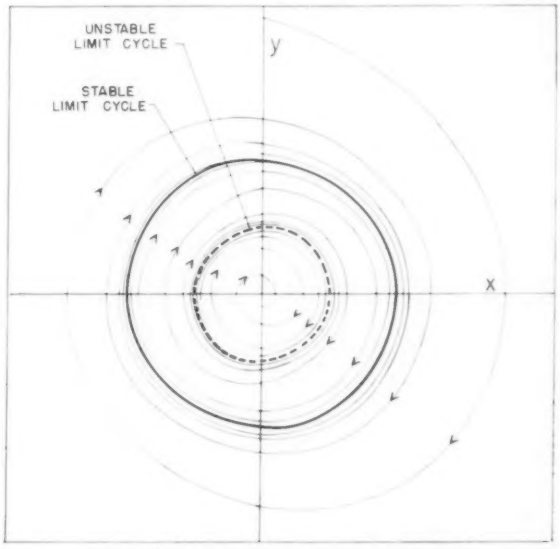


FIG. 18. Phase plane of a hard oscillator.

is that another degree of freedom is called into play and a jump occurs. Such phenomena are called relaxation phenomena. Usually such jumps occur in pairs and the system exhibits hysteresis. Mathematicians call these problems singular perturbation problems. We shall discuss them again in connection with partial differential equations. Relaxation phenomena occur in light switches, in structures that "oil can" and buckle, in certain involuntary muscular contractions, in electrical multivibrators, in flip-flop circuits used for counting and computing, in certain controls for automatic transmissions, and in various warning blinkers such as turn signals and road construction lights.

Behavior of Stimulated Autonomous Systems

At this point we turn our attention from isolated systems to systems that receive stimuli through their inputs. In such cases, our interest in a system increases in proportion as the output is unexpectedly related to the input. We saw that isolated linear systems are generally uninteresting. However, stimulated linear systems have a chance of competing with nonlinear systems in interest. Linear systems with small damping exhibit resonance in which, to the uninitiated, the response is unexpectedly great when compared to the input, as shown in Fig. 23. Let us see what a nonlinear system can do under similar circumstances. Suppose that the spring force of an oscillator is proportional to $x + x^3$ so that the stiffness of the spring increases with deflection. This will distort the resonance curves so that at higher amplitudes the resonance peaks are shifted to higher frequencies as shown in Fig. 24. Let us examine what happens if we gradually increase the frequency of the stimulating signal. At low frequencies, the response is similar to that of the linear system. As we increase the frequency, the response nears the bent-over peak of the curve, and we are faced with a now familiar dilemma. The frequency will continue to increase but the curve turns back on itself. Our system jumps from A to B with an

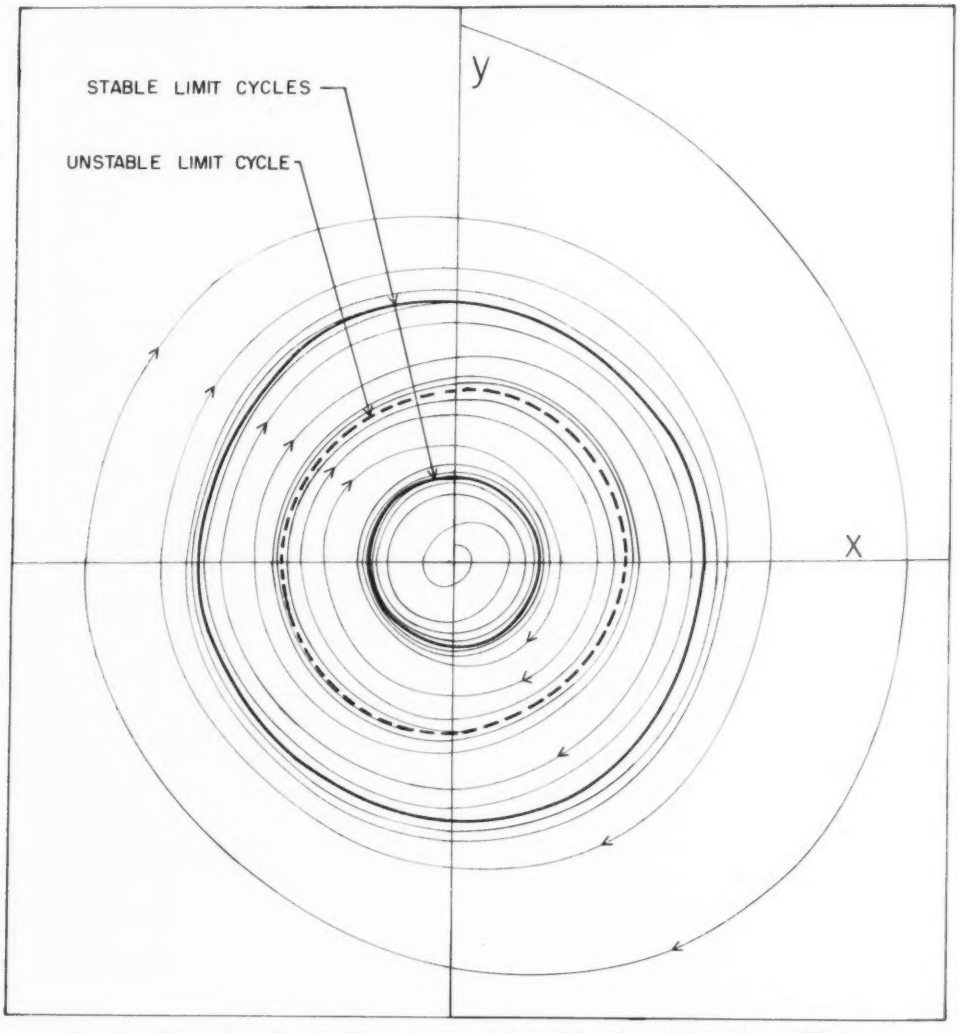


FIG. 19. Phase plane of an oscillator that can perform either of two independent oscillations.

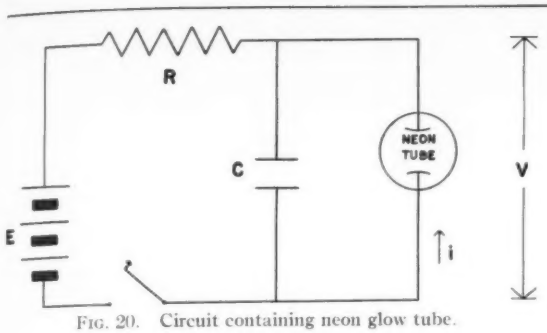


FIG. 20. Circuit containing neon glow tube.

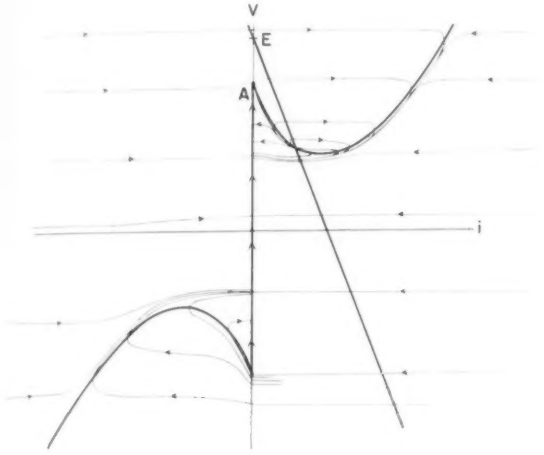


FIG. 22. Phase plane showing dynamic characteristic of neon tube circuit.

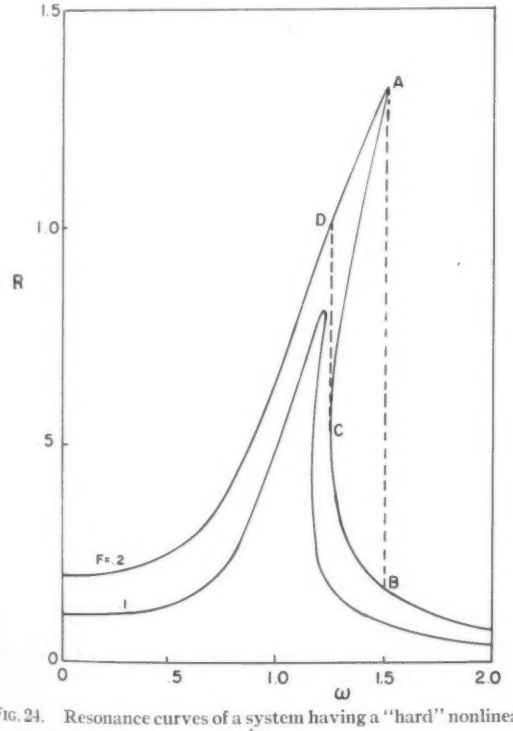


FIG. 24. Resonance curves of a system having a "hard" nonlinear spring.

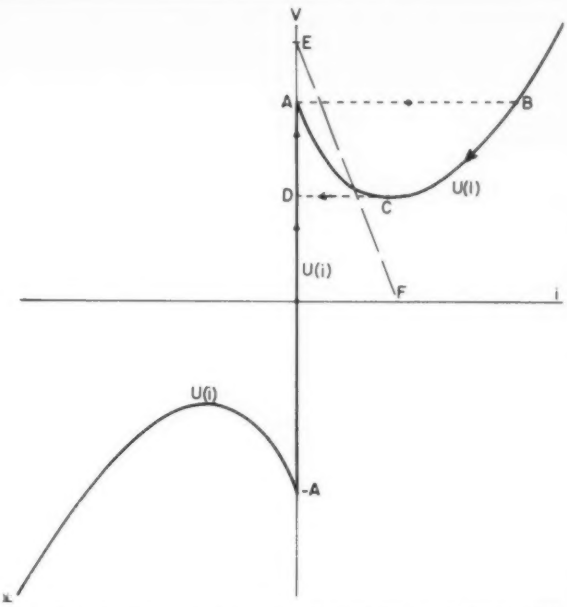


FIG. 21. Nonlinear voltage-current relationship of neon glow tube.

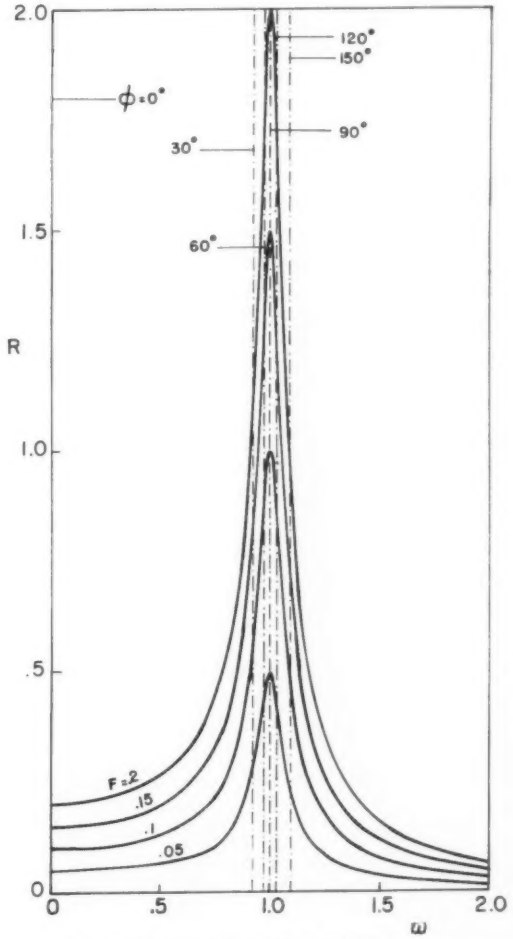


FIG. 23. Resonance curves of a linear system.

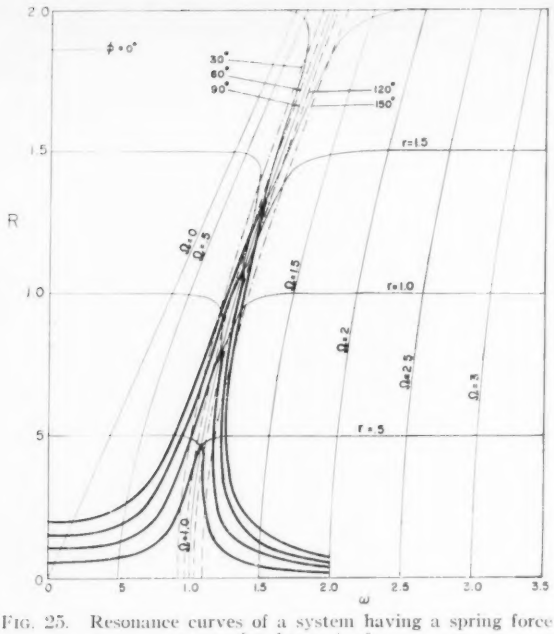


FIG. 25. Resonance curves of a system having a spring force proportional to $x + x^3$.

abrupt decrease in the output signal. It then continues outward along the curve through B. If the frequency is now decreased, the response does not jump from B to A, but from C to D. In a manner reminiscent of the flat arch and the neon tube, nonlinearity has added jump phenomena and hysteresis to resonance.

If R and ϕ are the amplitude and phase angle of the system's response to an input of amplitude F and relative frequency ω, then for a linear system with damping coefficient δ, these quantities are related by the equations $R(1 - \omega^2) = F \cos \phi$, $R\delta\omega = F \sin \phi$. For the nonlinear system, these equations become $R(1 - \omega^2 + 3R^2/4) = F \cos \phi$, $R\delta\omega = F \sin \phi$. If we distort the R, ω coordinate system by the transformation $\Omega = \sqrt{\omega^2 - 3R^2/4}$

$$r = R \sqrt{\frac{[(1 - \omega^2 + 3R^2/4)^2 + \delta^2\omega^2]}{[(1 - \omega^2 + 3R^2/4)^2 + \delta^2(\omega^2 - 3R^2/4)]}}$$

as shown in Fig. 25, then the response of the nonlinear system may be obtained by plotting the linear response curves of Fig. 23 directly on the distorted grid of r and Ω. If we had used a "soft" spring with force proportional to $x - x^3$, the resonance peaks would have been bent toward lower frequencies.

Now let us consider more general forms of analytic nonlinearity. A term of the form $(d^\alpha x/dt^\alpha) \cdot (d^\beta x/dt^\beta) \cdot (d^\gamma x/dt^\gamma)$ causes a distortion of the R, ω coordinate system similar to that for x^3 if $\alpha + \beta + \gamma$ is even, with the pattern of the distortion depending only on $\alpha + \beta + \gamma$. We shall investigate the cases where $\alpha + \beta + \gamma$ is odd later. Suppose that instead of multiplying three derivatives together to form a nonlinear term, we multiply n derivatives together. If n and $m = \alpha + \beta + \gamma + \ldots$ are both even, the term merely introduces higher harmonics into the response and biases the motion in one direction, but it has no primary influence on the resonance phenomenon. If n is odd and m is even, the R, ω grid is again distorted, with the frequency lines, for example, being bent according to the law $\Omega = \sqrt{\omega^2 - \lambda\omega^m R^n}$ where λ is a numerical factor depending on n and m. When m is even, the nonlinear term acts as a damping term which will be discussed later.

The effect of nonlinearity is not limited simply to modifying linear resonance. It introduces phenomena unknown in linear systems. Whenever two oscillating waves are multiplied together, higher harmonics are produced; so it is at once apparent that nonlinearity should introduce the possibility of resonance at higher harmonics of the impressed frequency, and this is indeed the case. For example, a dynamic system having the spring force law $x + x^3$ will resonate when the third harmonic (or any higher odd harmonic) of the input signal coincides approximately with the fundamental of the system. If the system has a response amplitude and phase of R_1 and ϕ_1 at the fundamental of the input signal and R_3 and ϕ_3 at the third harmonic, these quantities satisfy the equations

$$R_1[1 - \omega^2 + (3/4)R_1^2 + (3/2)R_3^2] = F \cos \phi_1 + (3/4)R_1^2 R_3 \cos (\phi_3 - 3\phi_1)$$

$$R_1\delta\omega = F \sin \phi_1 - (3/4)R_1^2 R_3 \sin (\phi_3 - 3\phi_1)$$

$$R_3[1 - 9\omega^2 + (3/2)R_1^2 + (3/4)R_3^2] = (1/4)R_1^3 \cos (\phi_3 - 3\phi_1)$$

$$3R_3\delta\omega = (1/4)R_1^3 \sin (\phi_3 - 3\phi_1)$$

The first pair of equations are for the response at the fundamental frequency of the input signal and the sec-

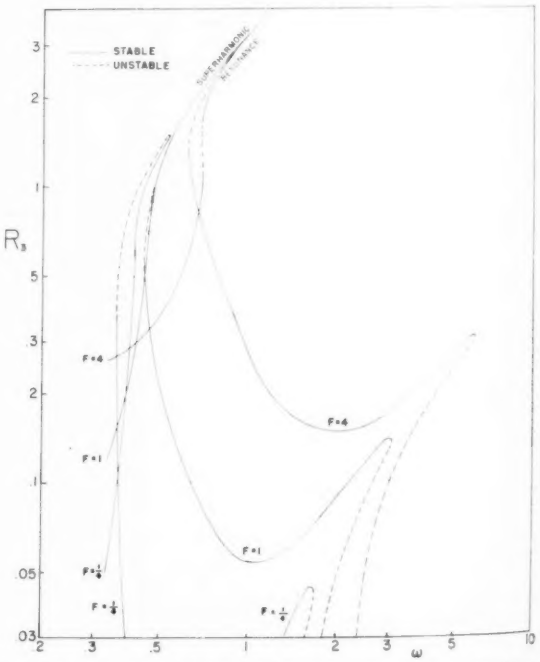


FIG. 26a. Response of the third harmonic for a system having a spring force proportional to $x + x^3$.

ond pair are for the third harmonic. The solutions of these equations have been plotted in Figs. 26a and 26b which show that nonlinearity not only bends over the resonance peaks of R_1 and induces minor resonance peaks in R_3 near $\omega = 1$, but near $\omega = 1/3$, it induces large resonance peaks in R_3 and completely changes the behavior of R_1. If we had included terms in R_5, R_7, etc., this same general pattern would have repeated itself at $\omega = 1/5, 1/7$, etc.

When we look at the equations used in the above discussion of superharmonic resonance, we are tempted to make the following speculation. In the second pair of equations, there are terms in R_1^3 which provide a driving force for the third harmonic whenever the fundamental is excited. In the first pair there is a related set of terms that gives the reaction of this third harmonic on the fundamental. Might it be possible for the coupling to work in a reverse manner—that is, might the fundamental excite a resonant response at a subharmonic frequency? If we investigate this possibility, we are led to the following set of equations.

$$R_{1/3}[1 - (\omega^2/9) + (3/4)R_{1/3}^2 + (3/2)R_1^2] = (3/4)R_{1/3}^2 R_1 \cos(\phi_1 - 3\phi_{1/3})$$

$$(1/3)R_{1/3}\delta\omega = -(3/4)R_{1/3}^2 R_1 \sin(\phi_1 - 3\phi_{1/3})$$

$$R_1[1 - \omega^2 + (3/2)R_{1/3}^2 + (3/4)R_1^2] = F\cos\phi_1 + (1/4)R_{1/3}^3 \cos(\phi_1 - 3\phi_{1/3})$$

$$R_1\delta\omega = F\sin\phi_1 + (1/4)R_{1/3}^3 \sin(\phi_1 - 3\phi_{1/3})$$

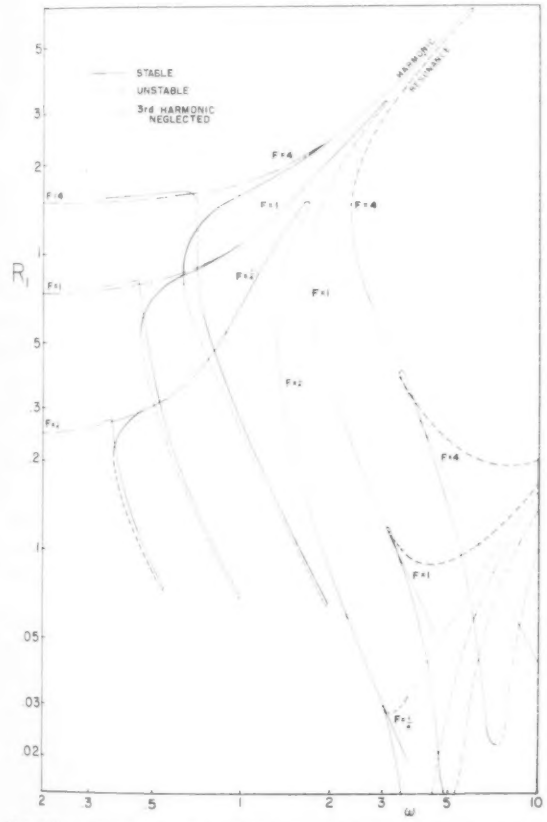


FIG. 26b. Response of the fundamental for a system having a spring force proportional to $x + x^3$.

Their similarity to those for the superharmonic response is immediately apparent. The solutions to these equations have been plotted in Figs. 26b and 26c, which show that a subharmonic resonance is induced near $\omega = 3$. The subharmonic response offers an alternate mode of oscillation rather than a modification of the harmonic response. Since the subharmonic response is unstable at the junction with the harmonic response, the transition to the subharmonic oscillation requires a threshold disturbance as in the case of a "hard" oscillator.

Similar resonances will occur at other subharmonic multiples of the fundamental frequency. It can readily be shown that subharmonic resonance will occur for the various types of analytic nonlinearity previously discussed as well as for a large class of topologically similar nonanalytic functions.

At this point we turn our attention to a different aspect of the effect of nonlinearity on resonant systems. Thus far we have considered only nonlinearities that were reactive in nature—i.e., those that acted as nonlinear springs, masses, inductances, etc. Now we investigate in a similar manner resistive nonlinearities. Interest here centers on the case where nonlinearity gives the system the capability of performing self-sustained oscillations as discussed in the section on isolated systems. The question is what happens when such a system is subjected to an external signal having a frequency near the natural frequency of the system. Instead of imposing an external signal, we may ask what happens when two nonlinear oscillators, tuned to neighboring frequencies, are mutually coupled together.

A system capable of oscillating by itself has three parameters that describe its performance—r_0 and ω_0, the amplitude and angular frequency of its oscillation, and τ, the characteristic time for the steady oscillation to be established. When the Van der Pol oscillator is described in these terms, its equation becomes

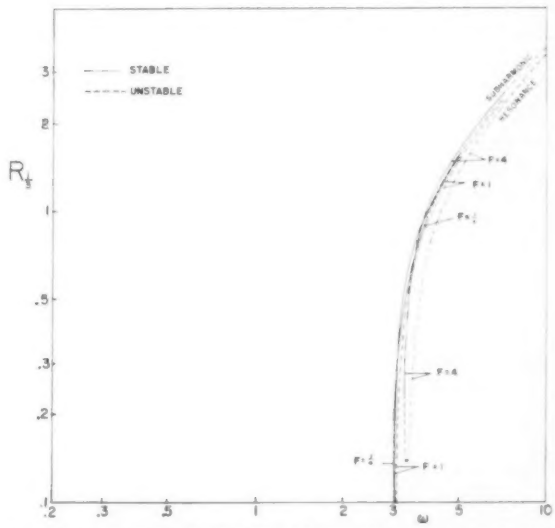


FIG. 26c. Response of the 1/3 subharmonic for a system having a spring force proportional to $x + x^3$.

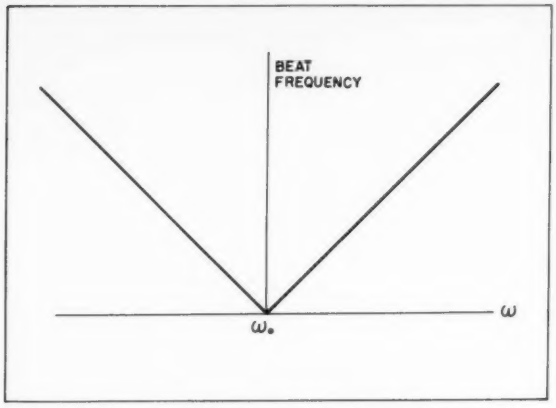


FIG. 27. Beat frequency of a linear oscillator.

$$\ddot{x}/\omega_0^2 - \tau\dot{x}(1 - 4x^2/r_0^2) + x = 0$$

If such an oscillator is subjected to an external signal, its equation is

$$\ddot{x}/\omega_0^2 - \tau\dot{x}(1 - 4x^2/r_0^2) + x = F \sin \omega t$$

In what follows we could equally well use the Rayleigh nonlinear term $4\tau\dot{x}^3/3\omega_0^2 r_0$ or any term of the form

$$\frac{\lambda\tau}{r_0^2\omega_0^{\alpha+\beta+\gamma-1}} \frac{d^\alpha x}{dt^\alpha} \frac{d^\beta x}{dt^\beta} \frac{d^\gamma x}{dt^\gamma}$$

provided $\alpha + \beta + \gamma$ is odd and λ is an appropriate numerical constant.

Let us see what would have happened if we had analyzed this problem from the linear point of view. First, we should have had to postulate an undamped oscillator in order to get it to oscillate steadily. Second, when an external signal is added, it would superimpose without influencing the existing oscillation and would cause the usual linear resonance phenomena. Near resonance, the two oscillations would produce a beat note, and this beat frequency could be made as slow as desired by making ω approach ω_0 as shown in Fig. 27.

Let us return to our nonlinear analysis. The primary response of the oscillator will be of the form

$$x = R \sin (\omega t + \phi) + R_0 \sin (\omega_0 t + \phi_0)$$

where R and ϕ are the amplitude and phase of the response at the frequency ω of the external signal and R_0 and ϕ_0 are corresponding quantities for the oscillation at the system's natural frequency ω_0. We readily obtain the following equations for these quantities.

$$R[1 - (\omega^2/\omega_0^2)] = F \cos \phi$$
$$\tau\omega R[1 - (R^2/r_0^2) - (2R_0^2/r_0^2)] = F \sin \phi$$
$$(2R^2/r_0^2) + (R_0^2/r_0^2) = 1$$

In Figs. 28a and 28b, the response amplitudes R/r_0 and R_0/r_0 have been plotted against the frequency difference $(\omega - \omega_0)/[\tau\omega\omega_0^2/(\omega + \omega_0)]$ for constant values of the amplitude of the input signal, $F/\tau\omega r_0$.

The behavior of the oscillator divides itself into two patterns. When $F/\tau\omega r_0$ is large, the response of R/r_0 to the external signal looks very much like a linear resonance curve. However, the amplitude of the natural oscillation, R_0/r_0, experiences a very interesting change. When $(\omega - \omega_0)/[\tau\omega\omega_0^2/(\omega + \omega_0)]$ is far from resonance, there is little influence of the external signal on R_0/r_0, but as ω approaches ω_0, the amplitude of R_0 drops to zero and stays there until ω recedes from ω_0 on the other side of resonance. Thus there is a finite range of frequencies near resonance where the oscillator will be locked or entrained so that it oscillates exactly with the frequency of the external signal. In this region, the beat frequency will disappear completely as shown in Fig. 29, which should be compared with the linear case of Fig. 27.

In a similar fashion, two coupled oscillators will lock in step when tuned to neighboring frequencies. Such frequency lock-in is of rather common occurrence. The sweep oscillators in television sets lock in with a signal from the sending station to keep the picture steady on the screen. When several alternators are connected to a power line, they lock together rather than cause beats in the line voltage. When two clocks are hung on the same wall, they fall into exact synchronism as do the multiple strings of a piano key. Lock-in occurs when the induced oscillation is approximately 70 per cent of the strength of the natural oscillation.

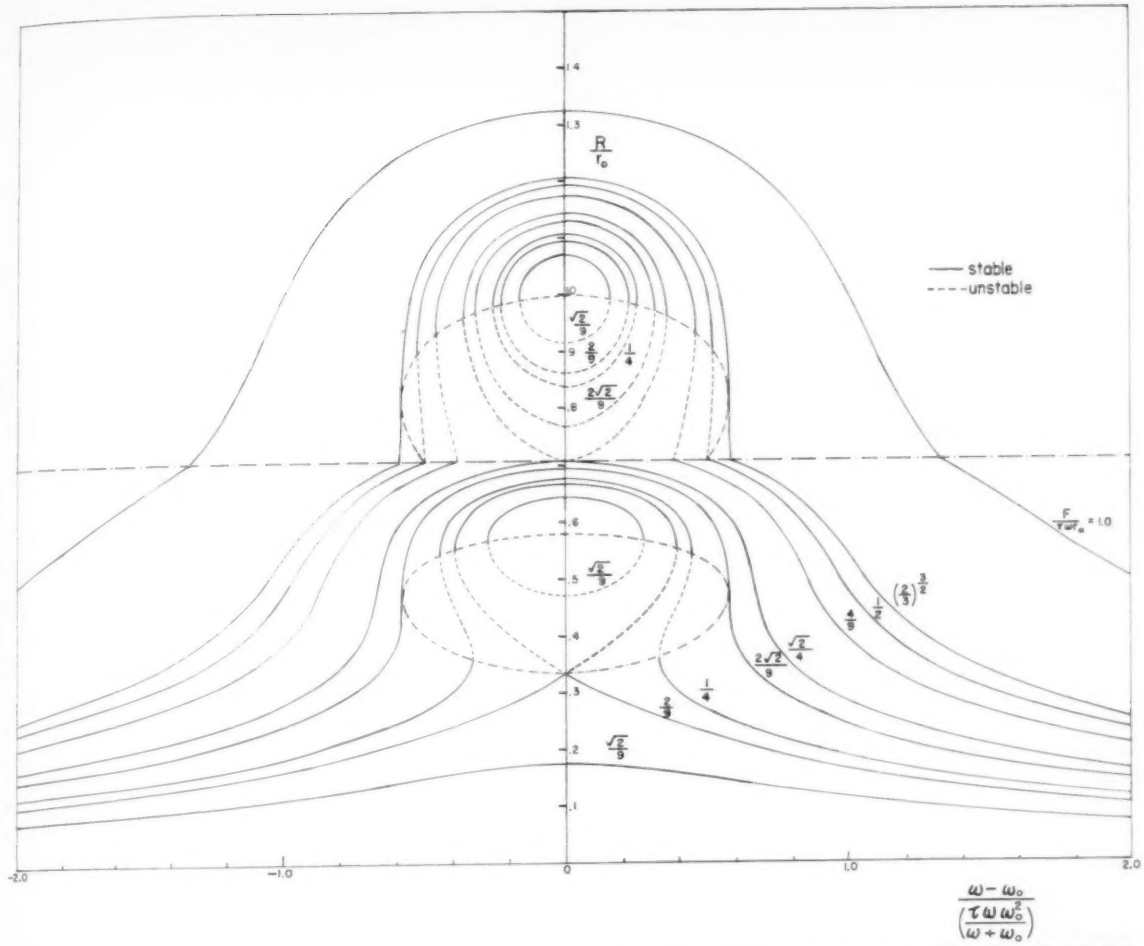
The oscillator exhibits a second pattern of behavior when $F/\tau\omega r_0$ is less than $(2/3)^{3/2}$ and $(\omega - \omega_0)/[\tau\omega\omega_0^2 \div (\omega + \omega_0)]$ is numerically less than $(1/3)^{1/2}$. Here, the response curves are multiple-valued and more than one stable-state of affairs exists, some with lock-in and some without. Hysteresis and jump phenonena occur. Such behavior is quite different from anything predicted by linear theory.

Behavior of Nonautonomous Systems

Consider for a moment the system represented by the simple equation, $\dot{x} + f(t)x = 0$. In our thinking, $f(t)$ can stand for either of two quite different entities. If $f(t)$ is fed into an input of the system from outside, then it has a status akin to an independent variable. Our interest is to determine the output, $x(t)$, for any $f(t)$, thus $x(t) = \exp [-\int^t f(\tau)d\tau]$. In this case, the system is nonlinear and the combined effect, $x_3(t)$, of two causes, $f_1(t)$ and $f_2(t)$, is not the sum of their individual effects, $x_1(t)$ and $x_2(t)$. The system is autonomous because its behavior does not depend explicitly on the time.

However, if $f(t)$ is generated internally, then it represents not any possible function, but a specific function generated by a clock or its equivalent within the system. This function is a characteristic part of the system which is now nonautonomous. In such a case, the system, $\dot{x} + f(t)x = 0$, is isolated and linear.

It is clear that the "clock" inside a nonautonomous system can cause the system to exhibit a spontaneous behavior of its own, even when nothing is fed into the inputs. Because of this internally generated behavior, the concepts of equilibrium and stability become meaningless and, except in special cases, must be abandoned when analyzing nonautonomous systems.



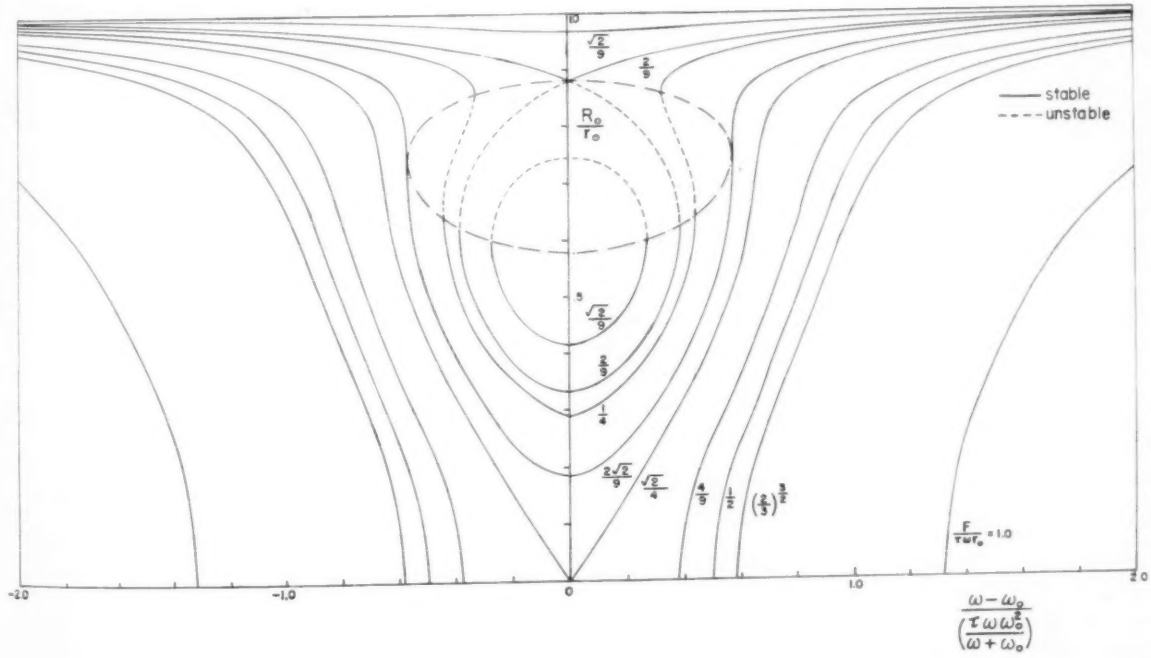


FIG. 28a (*top*). Amplitude of response at frequency of external signal. FIG. 28b (*bottom*). Amplitude of response at natural frequency of oscillator.

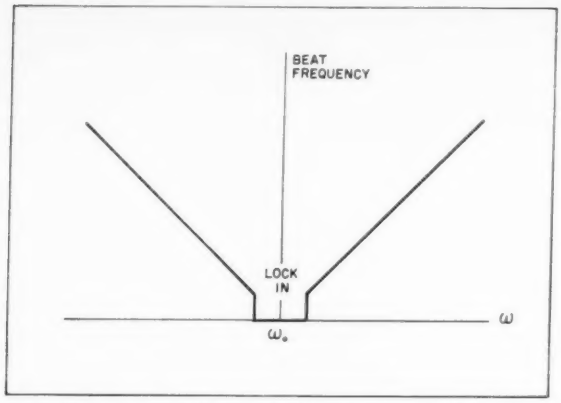


FIG. 29. Beat frequency of a nonlinear oscillator.

As an example, consider a guided missile which is closing on a target at a fixed rate so that there is a direct relationship between separation distance and time. The missile has a seeker head which observes the range and relative movements of the target. The missile uses these data to redirect its own path toward collision. Here we have a self-contained or closed-loop system whose characteristics depend explicitly on the range—i.e., the time to collision. Such a system is nonautonomous and we are unable to tell whether an increase in the violence of the missile's maneuvers as it approaches the target indicates instability or simply an increase in responsiveness.

In nonautonomous systems, we are usually interested in how a specific system's behavior depends on certain nonautonomous functions rather than a general picture of how various types of functions affect the system. A detailed discussion of nonautonomous systems would be quite lengthy, even for linear systems, and is outside the scope of the present paper.

Behavior of Systems with Several Independent Variables; Partial Differential Equations

Thus far we have discussed dynamic systems having only a single independent variable. Now let us consider what happens when the behavior of a system must be described in terms of several independent variables. Instead of considering a single continuous sequence of events in time or in space, we must now consider a multidimensional field of such events. Suppose that our system is a taut string whose response is given by the equation $\partial^2 y/\partial t^2 = a^2 \partial^2 y/\partial x^2$. The displacement of the string, y, as a function of x and t, is the output of our system. In Fig. 30, the string was plucked at A, and its resulting response is represented by a three-dimensional surface. At a later point, B, one end of the string was abruptly set in motion and the effect on the string's behavior is shown. In this geometrical picture, the input to the system comes through the boundaries of the system's response field. These input boundaries need not always be external boundaries as shown by the example in Fig. 31 for a thin airfoil in a supersonic stream. Here the response variable singled out for presentation is the magnitude of the velocity. The airfoil is an internal boundary which acts as an input to the system. In the case of a membrane subjected to a varying lateral pressure, the input may be applied throughout the field.

The problem of understanding the nonlinear behavior of systems involving partial differential equations is considerably more difficult than for those involving ordinary differential equations. The work that has been done on nonlinear partial differential equations represents more of a collection of solutions to problems than an attempt to abstract broad principles of behavior. Perhaps the best discussion of the efforts to understand the effects of nonlinearity in various fields is given by von Kármán in his Gibbs Lecture, "The Engineer Grapples with Nonlinear Problems."[6]

Behavior of Isolated Autonomous Systems

Here we may ask the same question that we asked about systems having a single independent variable—namely, can isolated autonomous systems exhibit an interesting behavior of their own? For linear equations, the mathematicians have been able to assemble an impressive list of theorems to the effect that if the response variable is zero on the boundary, it is zero everywhere in the field or, in our terms, the response of a linear system comes from its inputs and not from any self-generation in the field.

Although this question has not been explored extensively for nonlinear partial differential equations, we know the answer is quite different. For example, consider the case of a blunt cylindrical object in a flowing fluid having a moderate Reynolds Number. The flow will begin to oscillate spontaneously and a vortex trail will be shed periodically as shown in Fig. 32. Far downstream, remote from any boundaries, the vortex trail settles into a periodic pattern as shown in Fig. 33, a phenomenon which was first discussed by von Kármán.[9, 10]

A second example is the case of a boundary layer developing on a solid wall. Tollmien[11] and Schlichting[12] showed that at a critical Reynolds Number, such a boundary layer will become unstable, and if slightly

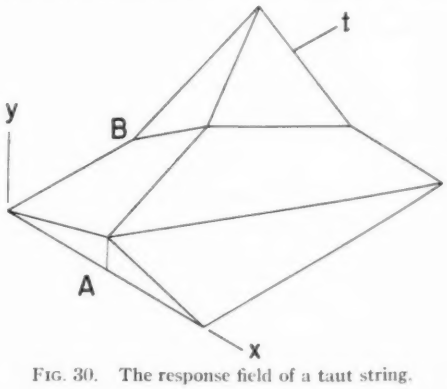


FIG. 30. The response field of a taut string.

disturbed, will begin to oscillate and, through nonlinearity, will become turbulent and exhibit a persistent random motion whose amplitude is self-stabilizing.

Additional examples may be found in the singing or galloping of wires and cables in a steady wind and in the oscillations induced by combustion in a rocket motor or by a heated grid in a Rijke tube. A beautifully illustrated example is that of the spinning detonation in a tube as seen in Fig. 34 which is a picture from Bone, Fraser, and Wheeler.[13] In some of these examples, the oscillatory behavior is controlled by a nonlinear coupling or feedback through the boundary, while in others it may come from the equivalent of nonlinear damping terms in the field.

We know in a general way why most of the above systems behave as they do, but in none of them can we say that we truly understand the nonlinear concepts involved.

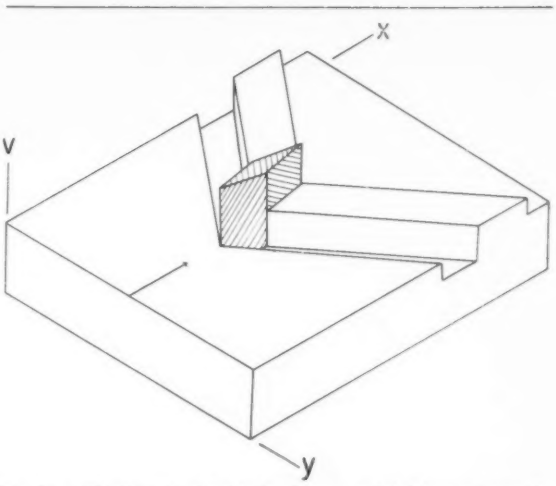


FIG. 31. The linearized field of a diamond airfoil in a supersonic stream.

Behavior of Stimulated Autonomous Systems

For stimulated systems, interest lies in the ways that such systems respond to stimuli fed in through the boundaries in the space of the independent variables. Here we shall focus attention, not on solutions with particular boundary conditions, but on the general response characteristics of various systems. It will be seen that, in many cases, we specify a system by giving a fixed set of conditions for a portion of the boundary and the inputs come through the remainder of the boundary.

Consider the movement of flood waters in a river such as the Mississippi. If x is the distance along the river, q the flow rate, and k the amount of water in the river per unit distance, then the equation of continuity at any point is $\partial k/\partial t + \partial q/\partial x = 0$. Seddon[14] has shown that for the movement of flood waters (as contrasted with more local dynamic behavior) the flow, q, is a function of the stage or flood height of the river which, in turn, is a function of k. This means q is related to k, a relationship which may vary from place to place in the river and from time to time, thus $q = q(k,x,t)$. Using this relation, the differential equation becomes $\partial k/\partial t + c\ \partial k/\partial x = a$ where $c = \partial q/\partial k$ and $a = -\partial q/\partial x$. This problem has been treated by Lighthill and Whitham[15] with quite interesting results.

Let us suppose that a solution of the above equation has been plotted in k,x,t space, and we select a given point on the solution surface for examination. There exist two vectors at this point whose components are $1,c,a$, and $\partial k/\partial t, \partial k/\partial x, -1$. The first vector's components are the properties of the system. The second vector is the normal to the solution surface. The differential equation says simply that these two vectors are mutually perpendicular, which is equivalent to saying that all solution surfaces passing through this point must lie along the vector $1, c, a$, as shown in Fig. 35.

If we move away from this point, the different solution surfaces fan out except along the vector, $1,c,a$. Consequently, if we proceed along a line, $dt/1 = dx/c = dk/a$, which always points in the direction, $1, c, a$, all solution surfaces through the original point will stay together and follow this line. Since the same is true for any other starting point, we may visualize the t,x,k space as being filled, spaghetti-like, with characteristic lines whose local directions are $1,c,a$, and we may form a solution surface by singling out a collection of these lines that pass through an initial value line as shown in Fig. 36.

Physically, this means that when a disturbance given by the curve, I, II, III, enters the system through the initial value boundary, it will propagate out as waves along the characteristic lines to determine the behavior of the rest of the field. For example, if at a given point on the river, the progress of the flood movement from $t = a$ to $t = b$ is given by I, then this behavior will propagate downstream along the characteristic lines going out from I. The initial progress of the flood may also be specified by its distribution along the river at a given instant, as III, or as recorded by a moving observer along the curve II.

The behavior of the waves is given by the characteristic wave lines, $dt/1 = dx/c = dk/a$. The speed of the wave is $dx/dt = c$, and its change of amplitude is $dk/dt = a$. Since both c and a are variable, the wave experiences both horizontal and vertical distortion as it propagates.

Let us consider the very instructive case of a river for which the relationship between q and k does not change with x or t. We then have $\partial k/\partial t + c(k)\ \partial k \div \partial x = 0$ and our system is nonlinear and autonomous. A given point on a wave will stay at the same height and will travel with constant speed, but different points on the wave will travel with different speeds so that horizontal distortion will occur. For most rivers, the flow increases ever more rapidly as the river rises so that the wave speed, $c = \partial q/\partial k$, increases with increasing q or k. In general, flood waves move downstream faster than the speed of the river so that the flood moves ahead of the water that produces it. An

FIG. 32. Vortex trail shed from a cylindrical obstacle.[8]

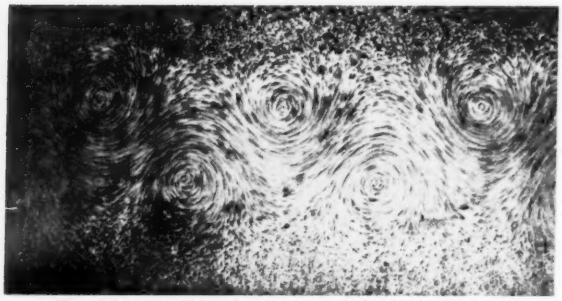


FIG. 33. von Kármán vortex trail far downstream.[8]

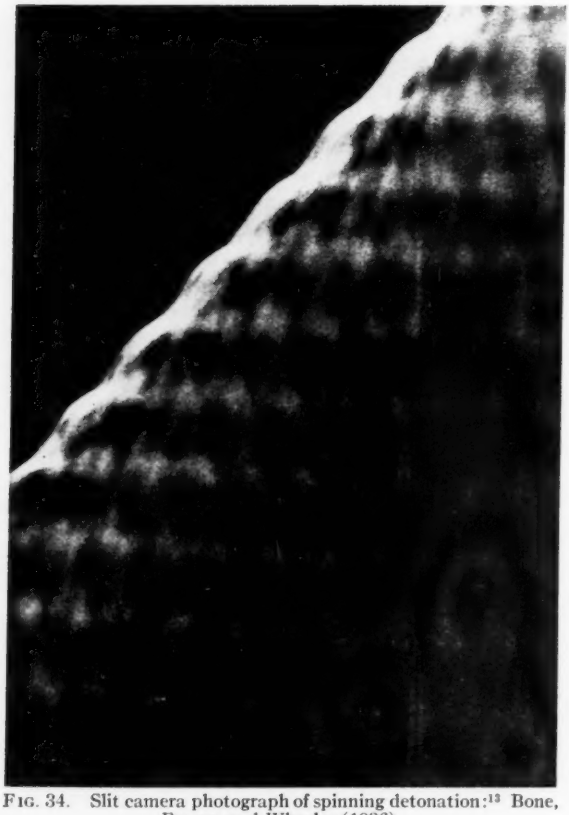


FIG. 34. Slit camera photograph of spinning detonation:[13] Bone, Fraser, and Wheeler (1936).

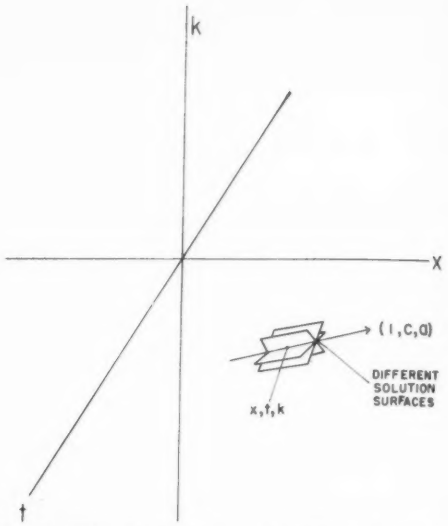


FIG. 35. Condition imposed by partial differential equation on solutions through the same point.

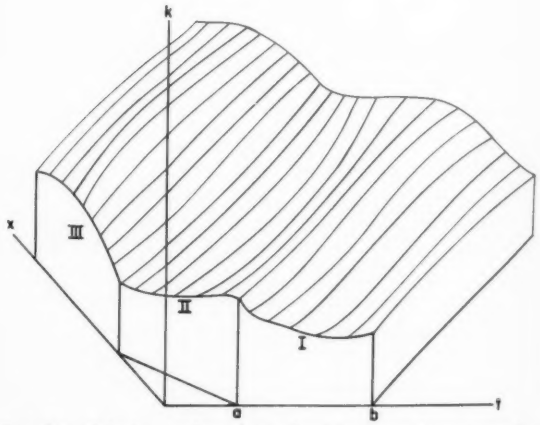


FIG. 36. Solution surface determined by wave lines emanating from initial value curve.

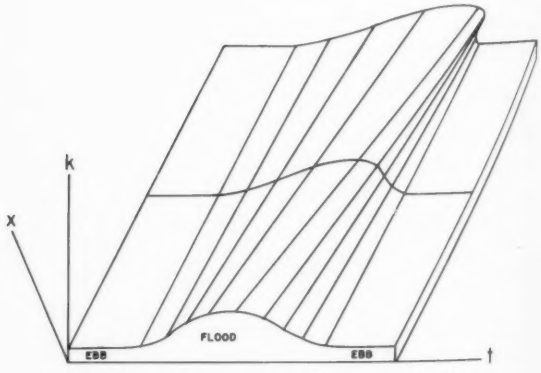


FIG. 37. Movement of flood wave down a river.

important effect of this wave phenomena results from the horizontal distortion of the wave shape. Consider the rise and fall of a flood as represented by the hump of Fig. 37. Because the wave velocity increases with increasing k, the flood crest will move faster than the ebb plateaus and the wave will steepen on the forward side and flatten on the rearward side as shown in Fig. 37. Our equation predicts that eventually the crest will ride out over the forward ebb region. Until now we have been viewing this phenomenon on a scale whose distances are measured in terms of hundreds of miles and whose times are measured in days. When a vertical wall of water starts to develop, our focus of attention must shrink drastically, and we see that it is necessary to take into account more complex dynamic effects which take place on a more rapid scale and which involve derivatives of a higher order. These terms are insignificant when viewed on the larger scale, but they intercede to control the steepness of the forward wall of the wave. A new balance of forces is established, and this flood front moves forward as a separate entity which has its own laws of behavior.

Suppose that the wave velocity, instead of increasing with wave height, decreases. In this case, the steepening occurs on the back side of the wave. Such a situation occurs in the case of the movement of traffic on crowded roads, an example also studied by Lighthill and Whitham.[16] These authors point out that when roads are lightly traveled, the traffic waves travel nearly as fast as the cars themselves; but as the traffic density increases, the wave speeds decrease to zero at the maximum capacity of the road; and for congested traffic, the waves travel backward. There is nearly a linear relation between wave speed and traffic density. Fig. 38, based on the work of Lighthill and Whitham, shows the wave pattern that occurs when a surge of traffic enters a highway. A forward traveling shock wave forms at the rear of the surge as drivers approach the crowded area and gradually pass through it. The figure also shows what happens when such a surge encounters a bottleneck sufficient to choke the flow. In this case, a rearward traveling wave forms, leaving a jam of cars crawling along the obstructed road. Eventually, the forward and rearward traveling waves coalesce and are swept through the bottleneck to clear up the jam.

The foregoing analysis does not predict the movement of any one automobile any more than the fluid flow laws predict the motion of a single molecule. However, it gives a remarkably accurate picture of the general flux of traffic. In many instances, if we are interested in the broad view of the behavior of a system and not in its local detail, then a first-order conservation equation of the form, $\partial k/\partial t + \partial q/\partial x = 0$, together with an algebraic relation between the variables k and q will suffice to describe the system. For example, consider the flow of a reactive fluid through a porous solid or packed bed such as occurs in chemical columns, silica gel driers, ion exchange resins, and chromatographic or electrophoretic separations. The reactants necessarily obey a conservation law, and if we are willing to view the process on a gross scale, we may assume the reaction is always in equilibrium. This leads to an algebraic relation between k, the concentration of one of the reactants, and q, the flow rate. When this relation is combined with the conservation law, we obtain $\partial k/\partial t + c(k)\ \partial k/\partial x$, the same type of first-order wave equation that represents the movement of floods and the flow of traffic. Again, the behavior pattern will consist of a single family of waves propagating through the system. Because of nonlinearity, through the dependence of the wave speed, c, on the concentration, k, such waves will necessarily steepen and form a concentration front moving through the system. Such fronts account for the sharp color demarcation that appears in chromatography and the steep concentration gradient that sweeps through a bed of silica gel or an ion exchange resin. When this front forms, we must abandon the assumption of continuous chemical equilibrium and include reaction rate terms. Goldstein[17] has done this with most interesting results, which are too extensive to be given here.

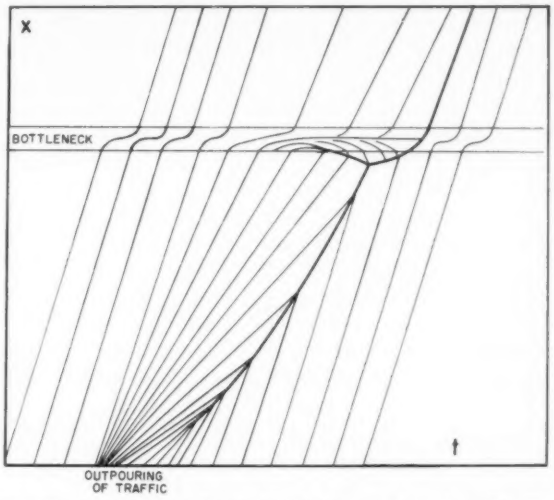


FIG. 38. Waves in traffic produced by an out-pouring of traffic and by a road bottleneck.

It can be shown that for *any* system which is described by a partial differential equation of the first order, the behavior is represented by waves propagating through the response field. As the nonlinearity of such equations becomes more complex, so do the ways in which the waves can distort. However, regardless of complexity, the general conclusion still remains that a system, which in the large view can be represented by a partial differential equation of first order, will have a response field consisting of waves which distort and produce local regions of special interest. These fronts or shocks will exhibit a behavior of their own, a behavior which can be analyzed only by considering terms of higher order. Such fronts, which are unknown in linear analysis, frequently overshadow in importance the smoothly varying remainder of the field.

The results that we have obtained for partial dif-

ferential equations of first order may be readily extended to equations of second order. If, in place of k in the equation, $\partial k/\partial t + c\partial k/\partial x = 0$, we substitute the expression, $\partial k/\partial t - c'\partial k/\partial x$, we have the second-order equation $(\partial/\partial t + c\partial/\partial x)(\partial k/\partial t - c'\partial k/\partial x) = 0$. In this case it is not k that is being propagated with velocity c, but a combination of derivatives of k. If c and c' are constant, we may write the equation in the alternate form, $(\partial/\partial t - c'\partial/\partial x)(\partial k/\partial x + c\partial k/\partial x) = 0$, which shows that by going to an equation of second order, we have introduced a second propagation phenomenon. Not only is $\partial k/\partial t - c'\partial k/\partial x$ propagated with velocity c, but $\partial k/\partial t + c\partial k/\partial x$ is propagated with velocity c'. Since the propagation of information about a set of derivatives is not sufficient to determine the behavior of the function itself, the remaining information must come from the other set of waves.

Consider the movement of an inviscid gas in a long tube, a problem that led Riemann to his investigation of wave equations. In this case, the density, k, and the flow rate, q, in the conservation equation $\partial k/\partial t + \partial q/\partial x = 0$ are not directly related, but are connected through their derivatives by the momentum equation. Riemann showed that the governing equation can be written $[\partial/\partial t + (c + v)\,\partial/\partial x]\,[v + \int cdk/k] = 0$ or alternately $[\partial/\partial t - (c - v)\,\partial/\partial x]\,[v - \int cdk/k] = 0$. In these expressions, $v = q/k$ is the flow velocity, and c is the local velocity of sound. The expressions, $v - \int cdk/k$ and $v + \int cdk/k$, represent different groupings of the derivatives of ψ where $k = \partial\psi/\partial x$ and $q = -\partial\psi/\partial t$. These groupings of derivatives are the quantities that are being propagated in the tube, one with velocity, $c + v$, the other with velocity, $-c + v$. From this combination of information coming from both ends of the tube, the complete picture of the behavior of the gas can be obtained. These equations, in spite of their complexity, were among the first nonlinear wave equations studied in detail. As is now well known, they correctly predict the steepening of compression waves into shock waves, a phenomenon which brings into play the higher order effects of viscosity and heat conduction to prevent the overturning which is impossible in longitudinal waves. For a more extensive discussion of this subject, the reader should consult such references as Courant and Friedrichs,[18] von Kármán,[19, 20] and the extensive work of Lighthill.

When a liquid such as water flows in a shallow stream, its equations are similar in form to those of a compressible gas, with the local depth playing a role equivalent to the density. Again, nonlinearity causes the waves to steepen and form hydraulic jumps in rivers or channels and breakers in the surf.

In meteorology, Tepper[21, 22] has shown that waves will form on an inversion surface in the atmosphere. With a frontal surface acting as a piston, such waves travel out into the air mass ahead of the front, break, and cause prefrontal squalls. Such waves have been observed to move one third the way across the United States. They have the anomalous behavior of causing weather to occur far out ahead of the front.

Suppose we have a quasi-linear equation of the form

$$A\,\partial^2\phi/\partial x^2 + B\,\partial^2\phi/\partial x\partial y + C\,\partial^2\phi/\partial y^2 = D$$

where the highest derivatives occur linearily and A, B, C, and D may be functions of x, y, ϕ, $\partial\phi/\partial x$, and $\partial\phi/\partial y$. This equation can be written $(\alpha\partial/\partial x + \beta\partial/\partial y)(a\partial\phi/\partial x + b\partial\phi/\partial y) = c$ where β/α and b/a are roots of the quadratic equation $A\lambda^2 - B\lambda + C = 0$. The factor, $a\partial\phi/\partial x + b\partial\phi/\partial y$, represents a "message" that is being transmitted as a wave in the "direction," $\alpha\partial/\partial x + \beta\partial/\partial y$, and c represents what is happening to the message during propagation. Now it may happen that the quadratic equation does not have real roots, as in the case of the Laplace equation, $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0$. Then there are no waves, and every part of the field is affected by all parts of the boundary inputs. A second possibility exists that the two roots may be equal, in which case the two waves coalesce as for the heat equation, $\kappa(\partial^2 T/\partial x^2) = \partial T/\partial t$.

Let us investigate the effect of nonlinearity on such elliptic or parabolic equations as they are called. In fluid mechanics, the equations for low-speed inviscid flows are elliptic in the space coordinates. For flows starting from rest, these equations reduce to the same linear equations that govern the behavior of electric and magnetic fields. However, in contrast to the agreement between theory and experiment in electricity and magnetism, actual flow patterns show wide divergence with the results of linear theory. It is for this reason that electricity and magnetism joined the ranks of the classical literature, while fluid mechanics still serves as the enchantress of nonlinearity.

The viscosity of common fluids such as air and water is so very small that it is difficult to see how it could greatly affect the flow of such fluids. By neglecting viscosity, we are led to the smoothly varying linear solutions of potential theory. However, here we have a situation that we have encountered previously in this paper. The viscosity is the coefficient of the highest derivatives in the equation, and even though we may neglect these terms in the larger view of the response field, there will inevitably be regions where reality can be maintained only by including the effect of these higher order terms. In wave equations, this occurred where nonlinearity caused the waves to steepen.

In potential theory, we choose solutions such that the fluid flows smoothly along the boundary walls. The viscosity, no matter how small, drags the fluid to rest at such walls. By erasing viscosity and reducing the order of our equations, we have lost the ability to impose the boundary condition of no slip at the wall. To correct this condition, we must alter our potential flow solutions by adding a thin boundary layer of viscous flow next to all solid surfaces. In this example for which the equations are elliptic, the critical region is not the steepened face of a wave but along all solid boundaries.

If the addition of this thin layer to all the classical potential solutions were the only act necessary to correct the situation, the life of an aerodynamicist would

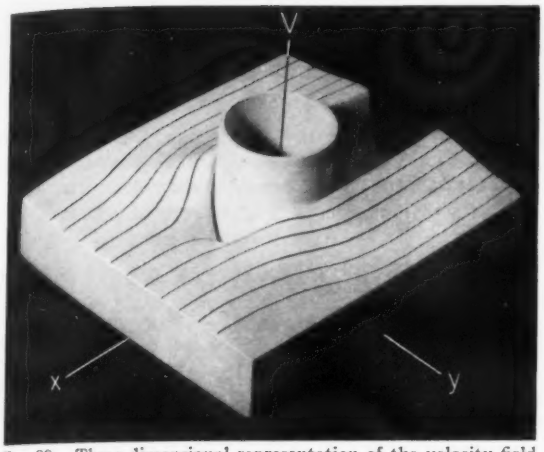


FIG. 39. Three-dimensional representation of the velocity field around a cylinder.

indeed be a simple one. However, nature is far more perverse and fascinating. The potential flow equations are linear, but this linearity depends on the flow being free from vorticity. The thin boundary layer introduces vorticity and with it nonlinearity. This enables the boundary layer to change its behavior pattern. The boundary layer can become unstable, begin to oscillate, break into turbulence, and develop a random motion which is self-perpetuating and self-limiting. Nor is this all. The layer need no longer serve only as a thin correction layer next to solid surfaces. Because it obeys a more complex set of equations, it takes on a character of its own, distinct from that of the larger linear field. Now it can widen out and cause major changes of its own in the flow pattern. This phenomenon, which we call separation, is of great technical importance, for it proves to be the limiting factor in the performance of all of man's fluid mechanical machinery. It determines the lowest speed of an airplane, the greatest thrust of a propeller, the highest pressure of a compressor. When separation occurs, we lose the power to determine the course of a flow by means of its boundaries. Instead, the viscous layer becomes the controlling factor.

Potential theory predicts a simple symmetrical pattern for the flow around an object such as a cylinder. Nonlinearity changes all this. In Fig. 39 is shown a three-dimensional representation of the flow that actually occurs. The height of the surface in this figure represents the magnitude of the velocity. The potential field is a gently undulating surface upon which the streamlines have been drawn. The boundary layers are sharp canyons that plunge to the foot of the cylinder. At separation, these canyons abruptly widen out to form the large trough of the wake.

CONCLUSION

At this point let us pause and take a broader look at some of the effects of nonlinearity that we have discussed. Here I should like to introduce an analogy that is both helpful and amusing. In much of the topography of the land around us, we see gently rolling hills and plains that are cut by rivers and streams as in Fig. 40. At such rivers and streams, the erosion process is more vigorous and the river banks are more precipitous than the surrounding countryside. Other anomalies in the erosion process leave cliffs, canyons, ridges and even whole mountain ranges scarring the face of the plain.

We have just seen in Fig. 39 that the solution surface for the flow of a fluid resembles a gently rolling plain with steep-walled boundary layers dropping sharply down to the edge of the solid surface. When the boundary layer separates, it widens out to dominate the topography as does a river when it reaches grade and widens out into an alluvial floodplain.

Nor does the analogy end here. In problems such as the flow of traffic on roads, the movement of flood waters in rivers, and the flow of chemical reactants through porous media, nonlinearity acts to flatten out the solution surfaces, concentrating all of the variations in height into cliff-like fronts. The plains-like portions of the solution surfaces are given by simple first-order partial differential equations. However, the steep fronts are governed by equations of higher order.

In supersonic flow, nonlinearity attempts to level out the flow field, not only by concentrating the velocity changes into thin shock waves that divide the flow field into regions of different levels, but also by forming boundary-layer canyons next to solid objects, with floodplain wakes trailing downstream.

In meteorology, the motions of the atmosphere are such as to form large air masses which are separated from each other by relatively sharp fronts. Within these masses, when interesting phenomena occur, they, too, tend toward a concentration of change within local regions. In oceanography it is now recognized that the boundaries of the great currents in the oceans are maintained sharp and distinct by nonlinearity, while the adjoining areas are relatively featureless.

We have seen that every partial differential equation has associated with it a characteristic family of ordinary

FIG. 40. Scene showing gently rolling countryside cut by the steep banks of a stream.

differential equations. By inverting this relationship, as Hamilton did, we may associate a partial differential equation with a system of ordinary differential equations. This leads us to apply the above analogy to relaxation phenomena in ordinary differential equations, where we again have the situation that the coefficient of the highest derivative is very small. A glance at the response curves of the neon tube of Fig. 21 shows that if a three-dimensional surface were constructed using the response curves as contours, this surface would have a gentle slope everywhere except in the immediate vicinity of the limit cycle and the characteristic curve of the neon tube. Here it falls away into deep canyons or rises in sharp ridges. The same situation exists for a multidimensional plot of the dynamic characteristics of the flat arch of Fig. 1. In such cases, a relatively simple equation governs the phenomenon throughout most of the field, and a more complex equation is necessary to describe the canyons and ridges. However, here we notice a difference from the examples given for partial differential equations. If we write our equation $\epsilon D = L$, where D represents the derivatives of highest order, ϵ the coefficient which is small, and L the lower order expression that remains when D is neglected, then in our examples of partial differential equations the plains-like solutions correspond to $L = 0$, while in the ordinary differential equations they correspond to $D = \infty$. It would appear that there might be problems in which the situations would be reversed.

We have examined many instances in which the effect of nonlinearity is to sweep into narrow regions of space and time the rapid changes in response that occur in physical systems and to leave behind broad expanses where change is slow and continuous. When we formulate a set of laws that suffices to describe the behavior of a system at one scale of the variables, nonlinearity creates narrow regions that require new laws for their explanation. If we examine these regions on a magnified scale, we frequently find that the process repeats itself, and we are faced with the choice of leading a perplexing but exciting pursuit of nonlinearity at front after front or of contenting ourselves with a calm, linearized exploration of the plains in between.

The eye of man is quick to discern the existence of regions of rapid change such as shock waves, boundary layers, and meteorological fronts. The mind of man goes even farther. These regions of rapid change become systems in their own right. The adjacent quiescent regions act as inputs to influence the behavior of the active layers. As an example, the turbulent boundary layer, which is particularly difficult to analyse theoretically, can be treated quite well experimentally as a nonlinear autonomous system. As the layer moves along a solid wall, it responds in a very recognizable manner to the varying pressure field of the outer flow and to changes in surface roughness at the wall. Even though the boundary layer continually grows, the present author[23, 24] has shown that it is possible to adopt a dimensionless set of response variables so that the boundary layer appears to be in equilibrium when the inputs, such as the nondimensional pressure gradient and the nondimensional shear gradient, are constant.

It may be possible to make more effective use of our large computing machines in such problems as weather prediction if we view the atmosphere as composed of nearly homogeneous masses separated by fronts which possess a behavior pattern of their own. In such an approach, it is necessary to devise a scheme that will recognize a front when it forms and predict its behavior after it has formed.

The exploration of the nonlinear properties of the world around us is as rewarding and challenging as it is difficult. There is every hope that it will lend reality to our analyses and satisfaction to our curiosity.

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Physics of Solids—Plastic Flow

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Introduction

The papers published by von Kármán in the fields of solid state physics and plastic flow represent a small fraction only of his scientific production. Each one of these papers, however, was an outstanding contribution which resulted in a better understanding of very fundamental physical problems. Following a chronological order, von Kármán's most important contributions were on the plastic flow of brittle materials under combined stresses, on the specific heat of solids, and on the propagation of plastic deformation in solids.

Plastic Flow and Fracture under Combined Stresses

Very early in his career, von Kármán was interested in the physics of plastic flow and fracture. In fact, his *Habilitationschrift* in Göttingen[1] was in the field of plasticity. The problem was to define the conditions under which a solid material starts to deform plastically.

Around 1900,[2] Föppl had studied the behavior of brittle materials under combined stresses. Cubes of rocks were loaded on two sides and tested to failure. The results indicated that the rupture strength was independent of the intermediate principal stress. The experiments made by von Kármán were the first in which specimens were subjected to external hydraulic pressure. The most complete set of experiments were performed on materials such as marble which are brittle under normal conditions. When cylindrical test pieces were subjected to uniaxial compression and simultaneous lateral pressure, these materials behaved very much like ductile metals. Curves similar to stress-strain curves for metals were obtained by plotting the difference between the principal stresses (axial and radial) versus the axial compressive strain. The amount of plastic flow and the bulging of the cylinder increased with increasing hydrostatic pressure. Twin markings within the crystals of calcite were observed, and it was quite clear that plastic flow took place through a mechanism similar to that found in ductile materials.

A few years after von Kármán's results were published, Boker[3] conducted similar experiments on marble and on cast zinc. In a high-pressure chamber similar to that used by von Kármán, he subjected cylindrical specimens to increasing axial loads under constant hydrostatic pressure and, in addition, studied the effect of increasing hydrostatic pressure under constant axial load. In the latter case, when the radial stress was larger than the axial stress, plastic necking of the cylindrical specimen was observed and von Kármán's findings were confirmed for both marble and zinc samples.

Many experiments have been performed since von Kármán's original work in an attempt to discover the rules governing the flow and fracture of solid materials under combined stresses and to formulate a general theory. In the same manner that brittle materials can flow plastically under certain stress fields, ductile metals and alloys may rupture without plastic deformation under certain conditions. Brittle rupture of mild steel in the case of welded merchant ships and of aluminum alloys in the more recent case of the Comet structural failure have emphasized the practical importance of the problem. In spite of the large number of investigations devoted to the subject of plastic flow and fracture, a comprehensive theory has not been formulated yet. The problem is obviously complex, and it is improbable that a satisfactory answer will be found by considering only the macroscopic mechanical properties of solids (such as stress-strain relationships, elastic limit, etc.) and mathematical theories based on assumptions derived from these bulk properties. The atomic nature of plastic deformation will have to be the foundation of a theory. The understanding of plastic deformation from an atomic point of view has made rapid progress during the last 20 years. The theory of dislocations has been very successful in describing the basic mechanisms of plastic deformation in crystals. The solid state physicist, however, is not yet ready to apply the theory to the complex engineering problems of flow and fracture under combined stresses, and the engineer, on the other hand, does not see clearly how to build a general theory of plasticity starting from the fundamental relations governing the motion of dislocations in very simple and idealized cases. The gap between solid state physics and engineering plasticity must be filled before a comprehensive theory of flow and fracture of solids can be formulated.

The Theory of Specific Heat of Solids

Thermal properties of solids such as the specific heat can be interpreted in terms of the vibration of the atoms around their average positions. This fundamental problem in theoretical mechanics did not fail to attract von Kármán, and the result was the famous Born-Kármán theory of specific heat published in 1912 and 1913.[4–5] On the basis of classical mechanics, the fact that the specific heat of solids approaches zero as the temperature approaches absolute zero could not be explained. Einstein[6] was the first one to give a semiquantitative explanation for the decrease in specific heat

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FIG. 1.

with decreasing temperature. By applying the quantum hypothesis to the allowed energy states of the oscillators and by assuming a Boltzmann distribution, he showed that the specific heat should decrease with temperature, but the theoretical results were not in satisfactory agreement with the measurements available at that time. Almost simultaneously, Debye[7] on one hand and Born and von Kármán on the other hand developed methods for computing the modes of vibrations of the atoms in a solid. The Debye theory received wide recognition because of its simplicity, and also because the agreement with measured specific heat values appeared to be quite satisfactory for several solids. It was soon recognized, however, that the Born-Kármán theory was more accurate. Instead of assuming that the spectrum of normal modes in a crystal is the same as in a continuous medium, as in the Debye theory, Born and Kármán established the equation of motion of the atoms and set up a method by which the frequency spectrum can be computed. One critical point in their derivation was to set up the boundary condition (known as the periodicity condition) which requires that the displacement of the atoms in equivalent sites on opposite faces of the crystal be the same. The theory was very successful in explaining certain anomalies in specific heat curves at low temperature. In principle the theory can be applied to any crystalline solid, but the computations become very involved when the crystal structure is not simple. Detailed computations have been carried out for cubic crystals (see, for example, references 8 and 9), and quite satisfactory explanations of anomalies in the specific heat vs. temperature curves at low temperature have been found. The Born-Kármán theory is, and will remain, a classical theory of fundamental importance in solid state physics.

The Propagation of Plastic Deformation in Solids

Early in 1940, von Kármán became interested in the problem of resistance of structures to impact loads beyond the elastic limit. In a conference held at the National Academy of Sciences under the Chairmanship of Dr. V. Bush, he conceived the idea of a plastic wave similar to an elastic wave but traveling with a lower velocity which might simply be $\sqrt{(d\sigma/d\epsilon)\ (1/\rho)}$ in which $d\sigma/d\epsilon$ is the slope of the stress-strain curve and ρ the density. In June, 1941, von Kármán asked the writer to set up experiments which would either prove or disprove the validity of this concept. With the help of Prof. D. S. Clark, experiments were performed at the California Institute of Technology. Copper wires, about 100 in. long, were impacted at one end at a constant velocity, and the impacting hammer was stopped after a time short enough so that the plastic deformation would not have time to reach the other end of the wire. The plastic strain distribution along the wire was measured after the test. Preliminary experimental results were discussed with von Kármán at the end of September 1941. Because he was leaving for an urgent pre-Pearl Harbor meeting in the east at that time, only a few minutes were spent on the subject. From his reactions, however, it was quite clear that the results were exactly what he had visualized. Two days later came a letter written on the New York Central and mailed in Schenectady on October 1, 1941, containing a brief but complete theory of propagation of plastic deformation in solids. This letter (pencil written) is reproduced in Fig. 1 with the thought that many of von Kármán's former students and collaborators will recognize in this document a typical example of the master's ability in reducing a complicated engineering problem to its most essential physical elements. The mathematical derivation is simple, straightforward, and elegant.

Because of wartime secrecy requirements, von Kármán's theory was not published until 1950.[10] At that time, extensive work on the generalization of the theory to bars of finite length had been made by von Kármán himself in collaboration with H. F. Bohnenblust, D. H. Hyers, and J. Charyk. The main content of these investigations has been recorded in a National Defense Research Council war report but has not yet been published in a technical journal.

Von Kármán's theory of propagation of plastic deformation in solids, like all theories, is based on certain assumptions which were clearly stated in his publication, namely: (1) The lateral contraction of the material—i. e., the contribution of the later contraction to the kinetic energy—is neglected; (2) The stress-strain curve is unique, and the material is not strain rate sensitive; (3) The slope of the stress-strain curve decreases continually with increasing strain. The first assumption is probably a valid one since the lateral contraction must represent a very small energy in comparison with the longitudinal extension. The second assumption is only an approximation for most metals since it is well known that the stress-strain relation is affected by the rate of strain. In fact, most of the discrepancies observed between experiments and theory have been attributed to a strain rate effect. Several attempts have been made to incorporate the strain rate into a theory of propagation of plastic deformation. One interesting approach was recently presented by H. E. Lee in a paper at the 1955 Symposium on Deformation and Flow in Solids, in Madrid.[11] The third assumption is valid for most metals except those which exhibit a well-defined yield point such as annealed mild steel. These materials are also extremely sensitive to strain rate. In addition, it is now well established[12] that under certain conditions of rapid loading up to a given stress above the static yield point, the strain remains elastic for a short time and then increases rapidly. To a first approximation, the time during which the strain remains elastic is a function of stress and temperature. This delayed yielding phenomenon has been the object of a number of recent investigations, and the atomic mechanism by which it takes place (through motions of dislocations) is now reasonably well understood. Because of the dependence of the strain, not only upon

(*Continued on page 468*)

Hydrodynamics—A Review of von Kármán's Contributions

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Introduction

The introduction to this portion of our birthday tribute to Dr. von Kármán may well be stated in terms of a flow analogy. At the beginning of this century, two gently meandering streams of knowledge, one of the perfect and theoretical fluid and the other of the viscous and empirical liquid variety, were vigorously infused at their streamlined and stable boundary by a jet stream receiving its substance from new ideas and new scientific tools and its impulse from the pressure of the aerodynamic age then rapidly arising. A very few outstanding men provided these basic and powerful new concepts and thought processes. Directly and indirectly, their contributions have ever since diffused mental momentum and matter into practically all stagnant and flowing portions of the original streams and have thoroughly changed and regulated their course into a common one of increasing depth in a bed of broader theoretical and physical concepts.

The ever-increasing load of problems this stream is required to carry by modern technology has been supported by the turbulent eddying and mixing of scientific thought, the generation of which can largely be traced to a core of correlated and distinct basic developments, amongst which Dr. von Kármán's research contributions have been most outstanding and sustained over half a century.

A review by J. C. Hunsaker[1] of what was known in hydrodynamics in 1915 was concluded by stating that even "inadequate theory, employed as a guide in a qualitative sense, is better than no theory at all; also there is always the possibility that, stimulated by the pressing demands of aeronautics, there may be important contributions to our conceptions of fluid dynamics, and that eventually what is now but half understood may be made clear."

Since that time von Kármán has clarified many of the then baffling problems and has guided much of the subsequent research toward sound and useful conclusions. It is, of course, not possible here to trace the implications of his writings, even in hydrodynamics alone, with any degree of completeness. The variety of problems involved necessarily imposes here a concentration on those few with which some degree of familiarity has been developed by the writer. In the following, therefore, a rather general discussion is given of the advances in five divisions of hydrodynamics, under appropriate headings, in which Dr. von Kármán has provided essential tools through his basic papers and specific services through engineering consultation.

Boundary-Layer and Turbulence Concepts

The extensive treatments of von Kármán[2,3] of all aspects of surface resistance and of the internal flow conditions created by the interaction of bodies and fluid flow are probably known most widely and are dealt with in detail elsewhere. It may suffice here to recall mainly the results of his researches with respect to applications in hydraulics and naval architecture, from which a large proportion of the experimental supporting evidence had come originally. The introduction of the similarity principle into the problem of turbulent flow brought the unification of concepts needed and placed the prediction of flow resistance and of power losses in closed and open channels on a comprehensive basis for hydraulically smooth and rough surfaces. The Kármán-Prandtl equations for these flow conditions have been combined empirically by Colebrook-White into a universal function for pipe flow covering also the transition ranges for statistically rough surfaces, which predominate in commercial products.[4] While this procedure is by now well-accepted in practice, it depends, however, on the empirical definition of a hydraulically effective roughness length (equivalent Nikuradse sand roughness) and cannot give a solution from a definition of mechanical roughness alone. The direct translation of the latter into the hydraulic equivalent is as yet not accomplished. Although von Kármán advised in 1934[3] that "one of the main objects of research (in this field) ought to be the standardizing of rough surfaces in some theoretically proper and practically useful way," progress has been confined only to the practical approach of evaluating the roughness from hydraulic measurement. Although a considerable number of investigations with various geometrically defined roughness patterns were made, correlation of hydraulic and geometric roughness quantities has not been achieved.

For open channels, the problem is complicated by the fact that the channel geometry changes with the discharge and thus the relative roughness becomes a variable. The method of determining, nevertheless, the effective roughness length was given by Keulegan.[5]

The application of boundary-layer theory to practical problems of nonuniform boundary resistance in hydrodynamics has given a number of useful results

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placing these on a basis of greater usefulness than the traditional catch-all coefficients of limited range. Such problems are, for example; the computation of flow factors for short tubes and nozzles, the entrance conditions and resistance losses for short sections of tunnels and pipes, extrapolations for scale effects for certain hydraulic models, flow capacities for broadcrested weirs, and the incipience of air entrainment for spillway flow.

The problem of flat plate drag is of obvious importance, particularly to naval architects, and affects vitally the prediction of skin friction for model and prototype ships, which cannot be assumed as very satisfactory at present. Following a fundamental reconsideration of the case of flat plates in zero pressure gradient by Landweber,[6] the need for data of greater precision for this purpose became very apparent.

The soundness of converting pipe flow data to analogous use with flat plates had been assumed by Prandtl and Schlichting,[7] who employed Nikuradse's data for rough surfaces for this purpose. It is readily apparent that the transition from a smooth to rough surface action may be obtained by transforming the Colebrook-White universal function in a similar manner, at least for practical purposes. Hama[8] in the latest treatment on the subject has proved that the transfer of pipe data to flat plate resistance is fully justified. However, the experimental work on flat plates has not proceeded to the point where transitions of the Colebrook-White definition could be produced with artificial roughness patterns, even when they were of a random nature. Thus the problem is still unresolved, as to what geometric surface parameters govern the hydraulic behavior in flat plate resistance.

The great importance of shore protection and of other marine engineering works has led to a revived interest in wave characteristics. The pertinent boundary-layer and turbulence relations are here introduced in the study of the damping of the waves as well as in the effect of transient boundary layers and of the inherent turbulence production on the movement of sediments and the consequent deformation of the sea shores. The measurement of a transient boundary layer formed under a solitary wave was recently accomplished by means of a suitable pressure cell–Pitot tube combination[9] and is shown plotted in Fig. 1 in dimensionless form by relating instantaneous velocities within the boundary layer to instantaneous velocities outside and corresponding to successive locations under the wave. It is believed that by such studies the complex phenomena of wave damping, sediment transport and sorting by wave action may eventually be rationalized.

The internal flow conditions of turbulent streams are largely described by the universal velocity distribution law which entails important consequences with respect to turbulent diffusion of matter and of other properties in conduits and channels. This was recognized early by von Kármán.[3] He stated the pertinent relations and suggested the first theoretical evaluation of sediment suspension and turbulence in open channels. This problem was most successfully attacked later by Vanoni[10] at the Soil Conservation Laboratory of the California Institute of Technology, analytically and experimentally, so that the size distribution of sediment carried in suspension by turbulent action is predictable for the two-dimensional stream. The damping of turbulence by excessive sediment loads and the capacity of turbulent streams to carry sediment, however, are still unresolved problems.

Incipient velocities for the movement of coarser granular sediment along the bed had been determined earlier by Shields (1936) on the basis of rational flow parameters, and his approach resulted in a consistent correlation of widely varying data. The application of turbulence concepts to the practical problem of sedimentation in settling basins has been carried out by Dobbins[11] and many other technical processes of diffusion have been approached since by corresponding reasoning. The reversed problem of entrainment of air downward from the surface of a swiftly flowing stream due to the action of turbulence is still being pursued without conclusive results so far, due to the more complex nonuniform action in this case.

Finally, the elegant solution by G. I. Taylor[12] of the diffusion of admixtures by a uniform flow along a conduit, which was based on the universal velocity distribution law and should prove of significant value in pipeline technology, must be referred to here. The problem of the dispersion of fresh water streams into the salt water estuaries is suggestive of a similar approach, but is vastly more complicated due to the periodic nature of the tidal flows, which act here as the turbulence producing dispersing agents, and due to the relatively large differences in density of the liquids involved, which in turn affect the internal flow conditions. Nevertheless, here also some progress will continue to be made on the basis of judicious application of rational analysis and of experimental procedures.

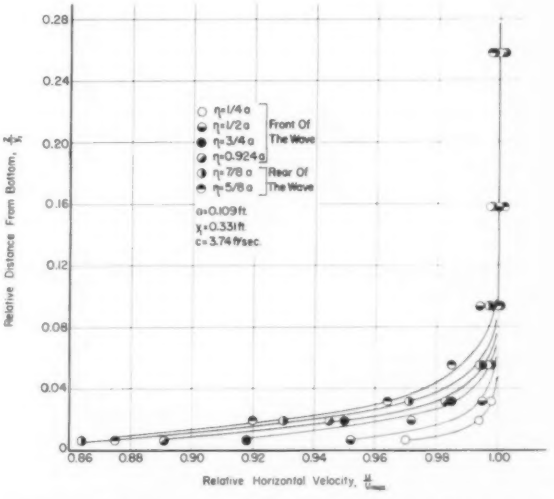

FIG. 1. Transient boundary layer under a solitary wave. a = amplitude, η = local surface elevation, c = wave velocity, y_1 = depth.[9]

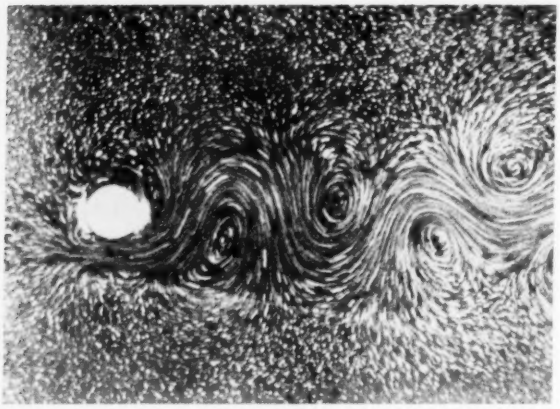

FIG. 2. Kármán trail behind a moving cylinder. (Prandtl-Tietjens.)

VON KÁRMÁN VORTEX TRAILS

In one of the earliest publications,[13] von Kármán and Rubach gave an analytical and physical explanation for the mechanics of form drag, which has become the basis of many important studies in this phenomenon thus demonstrated to be a dynamic one. Two rows of vortices with opposing sense of rotation were shown to be stable if shifted relative to each other by one-half the distance L between vortices in the same row and if the distance between the row centerlines was $h = 0.281\ L$. The vortex centers travel with a characteristic velocity u relative to the shedding obstacle and thus with a velocity $U - u$ with respect to the flow. The vorticity is determined by these quantities u, h, and L and, therefore, a relation could be given for the averaged drag coefficient as

$$C_D = [0.80\ (u/U) - 0.32\ (u/U)^2]\ (1/d)$$

where d is a characteristic length of the body. By obtaining the quantities u/U and L from observations of powder traces of vortex streets behind plates and cylinders in a water channel as shown in Fig. 2, Kármán and his collaborator were able to confirm the consistency of their analysis with the physical phenomenon.

The importance of this finding for the drag lies primarily in the definition of its character as dynamic rather than as static. Therefore, it has been of considerable help in the analysis of oscillations induced in bodies exposed to fluid streams. Many such problems have been traced to the vortex trail and have become of serious concern in technical applications whenever natural system frequencies were found close to the frequency of vortex shedding. Outside of aerodynamic applications, the significant *Studies of Vanes Singing in Water* by Gongwer[14] may be cited here as representative of many such phenomena, since they are of direct consequence to problems of hydrodynamically induced vibrations on ship propellers, hydraulic gates, and other hydraulic structures. His studies experimentally confirmed the excitation of vibrations by the vortex trail on strut sections that were moved by means of a rotating boom through a ring channel under a wide range of conditions. High-speed photography made the vortices readily observable through the cavitation induced in their cores. It was found that the transverse spacing of the vortices had to be adjusted by adding to the dimension of the trailing edge of the vane a distance related to the boundary-layer thickness in order to obtain the frequencies corresponding to the Kármán trail, so that $Nb'/U = 0.19$ (Strouhal Number) wherein b' is the adjusted edge thickness, N the frequency and U the speed of the vane.

This problem is also important in connection with hydraulic turbine practice and with the design of structural components for hydrofoils and submarines. Other technical applications in this field are concerned with the transverse forces exerted by wind action on high elastic stacks and on suspended wires and cables, but, on a more minute scale, such forces also affect the settling of sediment and the rising of air bubbles in water. A related hydrodynamic investigation by Meier-Windhorst[15] may be mentioned here, in which the Kármán vortex frequencies were systematically related to the natural frequencies of a submerged cylinder and in which the resulting oscillations were studied with and without damping beyond the Kármán frequencies. The similarity conditions were also derived for this general case of pendulations originating from hydrodynamic causes. This basic problem is of interest in connection with one of the major disasters of civil engineering practice—the destruction by aerodynamically induced oscillations of the Tacoma Narrows Bridge in 1940.

Von Kármán served as one of the three prominent engineers on the Board appointed by the Government to explore the causes of this disaster and supervised the analytical and experimental research concerned with the aerodynamic characteristics of this suspension bridge. These investigations contributed to a comprehensive reanalysis of suspension bridge design with respect to aerodynamic effects.[16] In later experiments in the wind tunnel by Farquharson, the torsional oscillations that wrecked the bridge were definitely traced to the Kármán trail.

IMPACT OF SEAPLANE FLOATS ON WATER

Following a visit to the United States von Kármán gave the first calculations of the impact loads exerted on seaplane hulls during landing in 1929.[17] He had learned of actual measurements of impact pressures in America and subsequently developed the pertinent formulas in his characteristic approximate manner, confining his analysis to V-bottom floats of small angle of dead rise. This permitted the assumption of the apparent mass as that of a flat plate at each instant of penetration into the water surface and the establishment of rather simple relations for the total load and maximum pressure intensities on the hull bottom. Estimates for falling due to wave action and for the limiting case of zero angle of dead rise, where the compressibility of the water must be considered, were also established. Later de-

velopments on the same problem by Wagner and others gave theoretical refinements such as consideration of piling up of water adjacent to the float and corrections for finite dead rise resulting in a considerable increase of apparent mass. Bisplinghoff and Doherty[18] summarized these theoretical advances and checked the various theories by drop tests of wedges for the two-dimensional case. A plot of maximum accelerations versus dead rise angle measured and obtained from the various theories is shown in Fig. 3.

With the revived interest in seaplanes for military purposes, considerable experimental work on planing surfaces and wave effects is now in progress to obtain data on three-dimensional effects and on specific models.

Impact forces of the type first calculated by von Kármán have become of considerable interest in naval architecture, where "ship slamming" has received increased attention. Szebehely and Todd[19] give the latest status report in this respect and deal with the pressure distribution on the bottom of a slamming ship on the basis of unsteady flow about the rapidly submerging bow. However, as is pointed out, the solutions are still rather approximate and the assumed mechanism of slamming is of necessity oversimplified in the analysis, which neglects effects of gravity, buoyancy, friction drag, etc., and is confined to certain simple geometric shapes. Some important conclusions have been derived by investigating the phase lag between bow motion and wave motion, the emergence of the forefoot of the ship, and the relative velocities between wave and bow. Knowledge of the relations between the unsteady pattern of slamming loads and the elastic response of the ship structure is essential before behavior in the rough sea can be much improved. This same conclusion holds for seaplanes, especially with respect to tip floats. A generalized treatment of this interdependency of wave configurations and elastic response, however, must await more detailed definitions of the statistical properties of the wave patterns of the open sea, a problem which at present is under intense investigation.

Hydraulic Analogy to Supersonic Motion

The standing wave patterns for supercritical flow in curved sections of open channels were systematically studied in 1935–1936 by Ippen and Knapp in connection with the high-velocity flows encountered in the concrete-lined flood channels of Los Angeles County. These phenomena were readily recognized by von Kármán as having their counterpart in supersonic gas dynamics and were thus accessible to treatment by analogous methods of analysis. These solutions were presented for the hydraulic case in reference 20 and resulted in a satisfactory correlation of the experimental data for flow in curved channels on the basis of the exact theory of frictionless fluid. Previous treatments of the analogy had been given by Jouguet and Riabouchinski and later by Preiswerk; the investigations mentioned above, however, led to the subsequent establishment of the complete theory for supercritical flow of water and its verification and application to a number of hydraulic problems, as well as to problems of transonic and supersonic aerodynamics.

Von Kármán had given the general relations for super critical two-dimensional flow for small transverse disturbances on the basis of negligible energy dissipation. This method lends itself to the construction of entire surface configurations by means of the Prandtl-Busemann hodograph method. In following up, the theory was also extended to include the case of finite disturbances with energy dissipation—i.e., of hydraulic shock waves or hydraulic jumps. These, however, are no longer related to their equivalent gas flow counterparts on the basis of mathematical analogy. Nevertheless, a useful analogy can be established on a purely physical and geometric basis, which is capable of giving qualitative and even quantitative results for many problems of gas dynamics. A summary of such problems was given by Harleman and Ippen.[21] Their analytical and experimental work covers many hydraulic investigations of supersonic airfoil lift, drag, and moment coefficients, as well as pressure distributions, with satisfactory quantitative results, even for the transient cases of shock wave diffraction about various obstacles. While, unfortunately, the hydraulic analogy can deal only with two-dimensional problems, it has nevertheless retained considerable importance in exploratory work for shock-tube and supersonic tunnel investigations and is an excellent tool of slow motion flow visualization for such work.

The applications of the small disturbance and of the

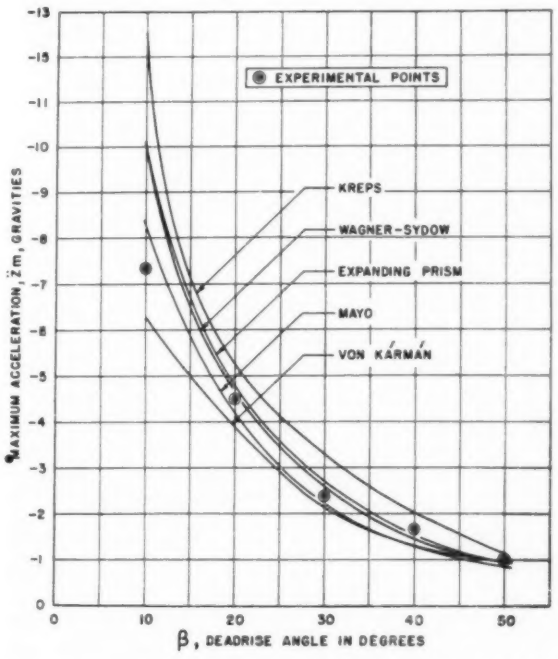

FIG. 3. Maximum acceleration vs. angle of deadrise for wedges dropping on water.[18]

shock-wave theory to various hydraulic structures such as channel curves, contractions, and expansions have been presented in a symposium,[22] from which Fig. 4 and Fig. 5 are reproduced. Fig. 4 shows the transverse oscillations of a supercritical flow below a channel contraction, while Fig. 5 gives the graphical correlation of all the variables entering into the two-dimensional theory of hydraulic shock or standing waves, with θ = deflection angle, β = shock-wave angle, F = Froude Number (analogous to Mach Number M), and h_2/h_1 = depth ratio (analogous to density ratio ρ_2/ρ_1). Thus, hydraulic engineering also has entered the "supersonic" stage, though by analogy, with many consequences with respect to the design of modern high head and high velocity structures.

Developments in Hydraulic Machinery

The connection of von Kármán's work with the field of fluid machinery is indirect as well as direct. In his fundamental paper about laminar and turbulent friction[23] he had included as one example the application of his boundary-layer relations to the case of a rotating disc, which gave for the first time a consistent theory for this problem. While disc friction, of obvious interest in connection with all turbomachinery as a source of headloss, is normally secondary, it has more recently become of major importance due to the trend towards higher speeds of rotation and the increased use of centrifugal pumps for liquids of high viscosities.[24] Von Kármán gave an analysis of disc friction, considering the development of laminar and turbulent boundary layers (the latter treatment based on the seventh root law of velocity distribution as introduced in that paper), and found it in essential agreement with experiments made in water. This treatment for the disc in an infinite fluid was complemented by Schultz-Grunow's analysis of the rotating disc in a cylindrical housing and by Goldstein through the application of the logarithmic velocity distribution law to the turbulent case. The coefficients of resistance given so far, however, cover only the case of turbulent smooth surface flow. The systematic treatment of disc roughness effects is presently under way at Massachusetts Institute of Technology.

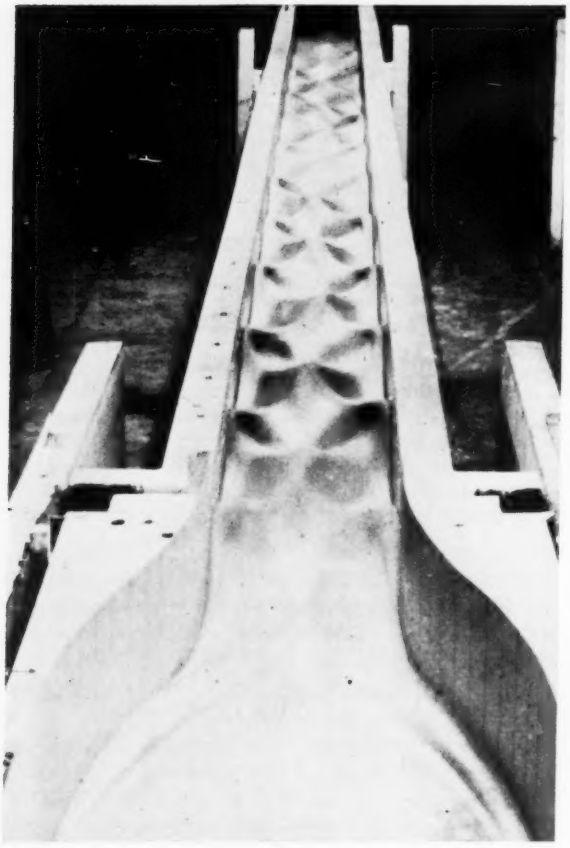

Fig. 4. Hydraulic disturbances in supercritical flow below contraction of channel.[22]

Von Kármán contributed indirectly to the field of turbomachinery by joining his colleagues, Professors R. T. Knapp and R. L. Daugherty, at the California Institute of Technology in 1934 in establishing and directing a hydraulic machinery laboratory for research on centrifugal pumps. Thus, a unique facility came into being which permitted performance experiments of unusual precision in connection with model tests for the large pumps for the Los Angeles Aqueduct of the Metropolitan Water District of Southern California. This laboratory later also carried on a similar successful program for the still larger pumps of the Grand Coulee Project of the U.S. Bureau of Reclamation. Subsequent additions of a Soil Conservation Laboratory and, during the war, of a Hydrodynamics Laboratory for Naval Ordnance at Caltech were undoubtedly influenced by the high record of performance achieved in these other research facilities. Many important contributions to the professional literature have come from the staff of these laboratories which have continued to provide research opportunities in the analytical and applied phases of hydrodynamics for over twenty years. Many of its former associates occupy leading positions now in academic, industrial, and government organizations.

Also, in connection with turbomachinery, von Kármán's affiliation as the chief aerodynamic consultant with a major wind turbine project may be cited here. The S. Morgan Smith Company, of York, Pennsylvania, together with P. C. Putman,[25] erected an experimental two-bladed windmill of 175 ft. diameter on top of Grandpa's Knob (2,000 ft. elevation) in Vermont, which stayed in intermittent service from 1941–1945. It was the first wind turbine which furnished sizable amounts of electric power to a commercial distribution system, and it operated successfully in winds from 17 to 70 miles per hour until the experiment was terminated due to a minor structural failure.

Concluding Remarks

Many of Dr. von Kármán's writings pertinent to hydrodynamics are dealt with elsewhere in this collec-

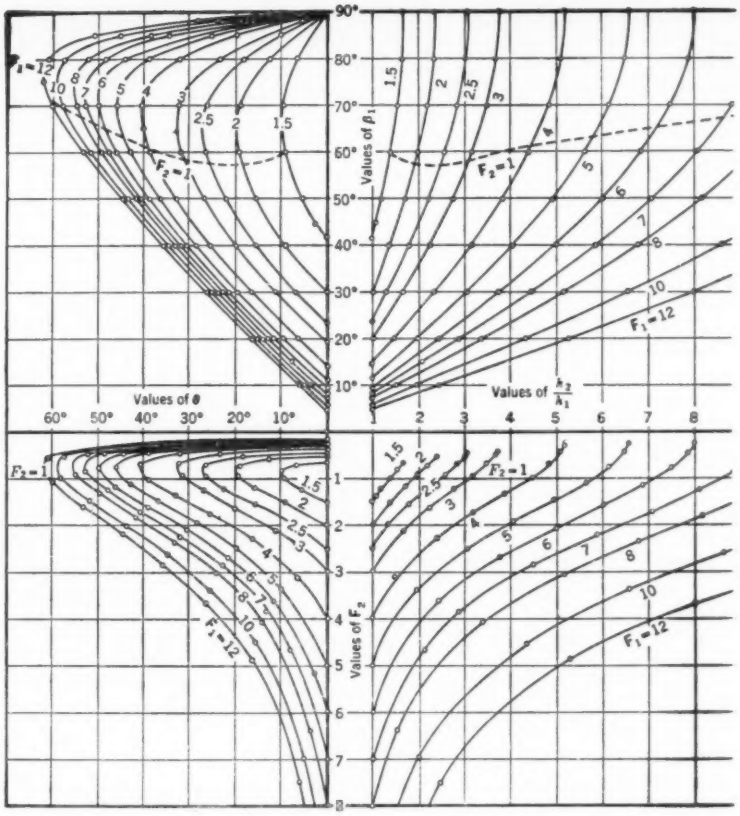

FIG. 5. General solution for hydraulic shock waves by two-dimensional theory.[22]

tive undertaking under special subject headings, due to their importance to all fields of fluid mechanics. In this respect the above account is, therefore, incomplete and confined to the limits somewhat arbitrarily set for the task on hand. It is hoped that, nevertheless, a satisfactory complement to this joint tribute has been achieved.

Much of the true nature of fluid motion has become clarified through von Kármán's life-long contributions. While technical applications often must still proceed on the cantilever of empiricism, the anchoring of our knowledge has been reinforced soundly and its span greatly increased through the many new physical concepts and mathematical formulations given in his papers. The skillful approach and the methods of analytical and of intuitive reasoning presented have affected much of the scientific writing of others and have penetrated into the technical literature of engineering teacher and practitioner. A proper assessment of such indirect influences is clearly impossible. It is equally impossible to summarize in any way the influence of his personal contacts on his countless students and associates and on their professional lives. This joint presentation may in a small way indicate that it is very persistent and is gratefully and deeply cherished by all.

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(*Continued on page 499*)

Some Features of Boundary Layers and Transition to Turbulent Flow

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SUMMARY

A brief review of the status of knowledge of laminar and turbulent boundary layers and of transition is given. Some new experimental results on transition in Poiseuille flow in a tube are reported. One set shows transition excited by the annular wake behind a ring airfoil. Another shows oscillograms of velocity fluctuations in the flow for several imposed disturbance amplitudes. These demonstrate successive stages in the breakdown of the flow.

(1) INTRODUCTION

THE RAPIDLY EXPANDING literature on boundary layers reflects the fact that there is a practical interest in an ever-widening range of Reynolds Numbers and in higher and higher Mach Numbers. The low Reynolds Numbers associated with hypersonic flight and the development of means of delaying transition have focused much effort on laminar boundary layers and on the mechanism of transition. Also, the drive toward a real understanding of the turbulent processes in shear flows, as well as toward the gathering of practical information on shearing stresses and interactions, has gained momentum with the refinement of experimental equipment and methods.

The space available here permits only a brief review of some aspects of boundary layers. Some of the topics are treated quite briefly, while others, notably heat transfer and the characteristics of turbulence in shear flows, have been omitted. These omissions are treated elsewhere in this issue in papers by Rannie on heat transfer and by Lin on turbulence.

More complete accounts of some of the topics and background material are given in references 1 and 2. Reference 3 describes briefly some recent investigations. A forthcoming volume of contributed papers on "Fifty Years of Boundary Layer Theory" was not available at the time the present account was written.

(2) LAMINAR BOUNDARY LAYER

Over the past decade particularly, theories of the laminar boundary layer have been extended into the hypersonic range of speeds, into the low density or slip flow regime, and into temperature ranges where gas imperfections must be considered. Shock-wave boundary-layer interaction has been the subject of many investigations. Three-dimensional boundary layers have likewise assumed a great practical importance, because of effect on the performance of axial flow compressors, on the flow over highly swept and low aspect ratio wings, and on bodies at angles of attack.

The original works by Busemann[4] and von Kármán,[5] on the compressible laminar boundary layer on a flat plate with Prandtl Number unity and viscosity proportional to a power of the absolute temperature were followed by that by von Kármán and Tsien,[6] in which heat transfer was taken into account. The most exact analysis to date on the flat plate is that by Crocco[7] for a range of Prandtl Numbers and several laws of variation of viscosity coefficient with temperature, including that due to Sutherland. An important result of these calculations is the relative insensitivity of the results to the law of variation of viscosity with temperature.

From a practical standpoint, important investigations by Howarth,[8] Illingworth,[9] and Stewartson[10] enable a comparison of compressible velocity profiles with "equivalent" incompressible profiles. Under the assumptions of Prandtl Number unity, viscosity proportional to absolute temperature, and zero heat transfer, they transformed the compressible boundary-layer equation into a form identical with the incompressible equation. Then, the transformations applied to any solution of the incompressible equation yield the "equivalent" compressible boundary layer. Crocco and Lees[11] applied the transformation to mixing between two streams with promising results.

The conditions under which similar solutions of the compressible boundary-layer equations[12–14] exist reveal further properties of the compressible boundary layer. With Prandtl Number unity and viscosity proportional to the absolute temperature, the conditions on the outside flow are derived such that distributions within the boundary layers are similar. By means of these equations, the effect of pressure gradient and heat transfer on skin friction and flow separation can be studied for specified pressure distributions. Solutions are obtained on an analog computer.[12]

Compressible boundary layers in continuous adverse pressure gradients are of practical interest at subsonic speeds, but at supersonic speeds the adverse gradients will, in most instances, take the form of incident shock waves. The details of the flow field in the vicinity of a shock impinging on a surface have been studied in detail by Ackeret, Feldmann, and Rott[15] and by Liepmann, Roshko, and Dhawan.[16] In the latter investigation, the authors found that, if the boundary layer is laminar, the flow separates in a limited region ahead of the impinging wave and the flow field is altered about 50 boundary-layer thicknesses upstream and several hundred thicknesses out from the surface. However,

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in spite of the flow separation and adverse pressure gradient, transition to turbulent flow did not always occur immediately aft of the shock. Donaldson and Lange[17] found a critical pressure ratio $\Delta p/q_1$ (q_1 is the dynamic pressure in the free stream) beyond which the laminar boundary layer separated for free-stream Mach Numbers of 1.6 to 2.05. They found the critical $\Delta p/q_1$ to be proportional to $R_x^{-1/3}$.† The shocks across which $\Delta p/q_1$ was measured were formed by a collar on a body of revolution.

Another type of shock-wave boundary-layer interaction occurs near the leading edge of a body in hypersonic flow.[18] Two influences are responsible for the interaction. First, the inclination of the shock wave is small at high Mach Numbers so it will lie close to the outer edge of the boundary layer, and second, the high temperatures generated by the deceleration in the vicinity of the leading edge cause the boundary layer to be many times thicker than it is at low speeds at the same Reynolds Number. As a result the shock will be detached and the curvature will be high at the apex. This interaction was first recognized by Tsien[19] and first investigated by Shen.[20]

A recent paper by Lees[21] goes into the details of this and subsequent investigations of hypersonic viscous flow. The parameter $M_\infty t/c$, where t is the thickness of the airfoil or maximum diameter of a body of revolution and c is the chord of the airfoil or length of the body, is Tsien's similarity parameter for inviscid hypersonic flow. However, for viscous flow, the interaction parameter $\chi = M_\infty{}^3\sqrt{C}/\sqrt{Re_x}$, where $C = \mu_w T_\infty \div \mu_\infty T_w$ (the subscript w referring to conditions at the wall), is found to govern the pressure distribution and the development of the boundary layer near the leading edge of a plate. The pressure is near ambient for low values of χ (far downstream) and increases linearly to several times ambient for high values of χ (near the leading edge). The effect of blunting the leading edge is to increase substantially the pressure for small x.

Analysis of the effects of high surface temperatures, such as occur in flight at high Mach Numbers, must include the effects of dissociation of the air and of variable air properties. Moore[22] has shown, on the basis of solutions on a differential analyzer, that over a wide range of plate temperature and heat transfer the effect of dissociation on the local skin-friction coefficient is negligible.

In flight at very high altitudes, where the mean free path is greater than about 1 per cent of the boundary-layer thickness, slip at the surface is encountered and air can no longer be treated strictly as a continuum. Tsien[23] defined the slip flow range as occuring when $0.01 < M/\sqrt{Re} < 0.1$ for $Re > 1$. Then the reflection and accomodation coefficients govern the energy and momentum of the reflected molecules, respectively, and hence the skin friction and equilibrium temperature of the surface. Although the trends of the experimental results on drag and heat-transfer coefficients follow theoretical predictions,[24] important discrepancies may be ascribed to shock-wave boundary-layer interactions near the leading edge and uncertainties in the values of the reflection and accomodation coefficients.

† Shock-wave interaction with turbulent boundary layers will be discussed in Section IV.

The investigation of three-dimensional boundary layers was greatly facilitated by the independent observations of Prandtl,[25] Sears,[26] and Jones[27] that the equations of motion for incompressible viscous flow past a yawed cylinder are separable. As a result, the components of the flow in the plane normal to the generators of the cylinder are independent of the angle of yaw—i.e., of the spanwise velocity component. This "independence principle" requires, however, that spanwise derivatives of all flow properties be zero. Therefore, transition to turbulence (see Section 3) as well as the development of the wake after laminar separation might be expected to be influenced by the yaw angle.

Simple methods of calculation are worked out by Rott and Crabtree,[28] and Crabtree[29] has given an approximate method for calculating compressible flow over a yawed cylinder, based on the Illingworth-Stewartson transformation. For supersonic flow past a yawed cone, Moore[30] and Hayes[31] showed that velocity and density in the boundary layer are constant along parabolas with vertices at the vertex of the cone and lying in a meridional plane. Moore analyzed the flow in the plane of symmetry for supersonic flow over a cone at high angles of attack. Differences in the behavior of the boundary-layer equations over different ranges of angle of attack were interpreted as indicating changes in the character of the flow over the lee side—i.e., boundary-layer flow at low angles of attack is supplanted by symmetrically placed vortices, and finally by the von Kármán vortex street at high angles of attack. Allen and Perkins[32] analyzed roughly the secondary flow over a cylindrical body and identified the drag due to the cross-flow component as an increment to the lift. They found excellent agreement with experiment. Mager[33] derived momentum equations applicable to three-dimensional flows including effects of rotation.

(3) Transition to Turbulent Flow

The stability of laminar flow, whether it be in a boundary layer, in a fully developed flow in a tube or channel, or in a jet or wake, is susceptible to analysis as a linear problem. A recent monograph by Lin[34] summarizes the state of the knowledge and shows that much is known about the stability of the basic flow to infinitesimal disturbances. However, the establishment of turbulent flow as a consequence of flow instability is far from well understood. The experiments of Schubauer and Skramstad[35] describe in detail the instability of the laminar boundary layer on a flat plate, but only very recently has it been possible to delineate some of the details of the changeover from regular to turbulent fluctuations. Some of these experiments will be described later. A study of transiton at supersonic speeds has recently been carried out by Probstein and Lin.[36]

There appears to be no single sequence of events that describes the transition phenomena under all circumstances. The initial disturbances may be two- or three-dimensional; they may be excited by turbulence,[35] by surface roughness,[37] by disturbances imposed within the boundary layer, or by shock waves.[17] Whatever their source, however, the rate of amplification, and therefore the rapidity with which transition follows the disturbance, will be affected by pressure gradient,[35] Reynolds Number,[35] Mach Number,[38, 39] suction,[40] heat transfer,[41] and curvature of the streamlines.[42–45]

If the initial disturbances are small, their subsequent growth or decay follows the linearized theory initially. If the disturbance grows, its rate of growth determines at what stage of its development the nonlinear terms, which were neglected in the analysis, take over and govern the transition process. If, on the other hand, the initial disturbance is large enough, the process may be nonlinear from the start.

Transition in three-dimensional boundary layers introduces some new aspects tending in general to move the transition upstream. A theoretical and experimental investigation by Gregory, Stuart, and Walker[45] was initiated after observations on swept wings showed transition occuring upstream of its position on unswept wings. The theoretical study concerned the stability of flow in the boundary layer on a rotating disc; this flow was chosen because it represents a relatively simple three-dimensional solution of the Navier-Stokes equation[46] and it shows some similarity with the boundary layer on a swept wing. The stability analysis showed a disturbance which may be described roughly as a superposition of Tollmien-Schlichting waves on Taylor-Goertler vortices.[42] The Taylor-Goertler vortices occur in a boundary layer over a concave surface and are known to exert a powerful influence on transition. Hence, their appearance in a three-dimensional boundary layer probably accounts for the earlier transition due to sweep. Experiments with china clay on the surface of a sweptback wing show streaks roughly parallel to the streamlines near the surface. The presence of these streaks is interpreted as indicating the presence of the Taylor-Goertler vortices, since their axes lie along the streamlines. Experiments at supersonic speeds[47] show qualitatively the same effect of sweep on transition—i.e., the transition point moves upstream with increasing sweep.

Present knowledge of the detailed nature of transition consists to a large extent of isolated observations on a small number of configurations. Visual studies of the life history of particular disturbance configurations such as horseshoe vortices[48,49] give a qualitative idea of a mechanism by which energy can be transferred from the main flow into fluctuations. However, it does not appear possible to describe by means of a single mechanism all of the phenomena associated with transition—e.g., the sudden changes that occur at the transition front.

These sudden changes are described phenomenologically by Emmons[50] postulate that transition comes about through the random appearance of "turbulent spots" and their subsequent growth. The rate of growth of the spots as observed on the surface of water in a shallow channel was found to check roughly with the measurements of Charters[51] on transverse contamination. By postulating that the spots appear in a random fashion in time and space and that a fully developed turbulent boundary layer exists within the spot, Emmons could match the observed dependence between average skin friction and Reynolds Number through the transition region.

The upstream location of the transition front would, according to Emmons' postulate, wander in a random fashion with time. The random wandering of the transition front at supersonic speeds has been observed by Evvard, Tucker and Burgess[52] by means of schlieren pictures of conical flow, and by Schubauer and Klebanoff[53] by means of hot-wire measurements of the intermittency factor on a flat plate at low speeds. The separate spots have been observed in schlieren photographs of supersonic flow as a laminar region sandwiched between two turbulent regions.

Schubauer and Klebanoff measured the details of the growth of the spots. They conclude, in agreement with Emmons, that transition occurs naturally as a succession of turbulent spots which grow more or less independently as they move downstream, finally merging into a turbulent boundary layer.†

Some measurements made recently by R. J. Leite[54] under the author's supervision add some information on the details of transition.‡ The measurements were made with a hot-wire anemometer in fully developed laminar flow in a circular tube 1.25 in. in diameter and 73 ft. (700 diameters) long. Air was discharged from a high-pressure source through baffles, screens, and a settling chamber, thence through the tube. About 68 ft. (650 diameters) from the entrance a disturbance generator, consisting of a loud speaker coil with axis parallel to the flow, was connected with an axially symmetric member inside the tube. The member inside the tube was oscillated axially at frequencies up to 100 cycles per sec. Reynolds Numbers $U_{max}a/\nu$, where a is a radius of the tube, varied from 4,000 to 13,000.

The interesting feature of this flow lies in the theoretical prediction that it is stable to all infinitesimal axially symmetric disturbances at all Reynolds Numbers.[55] Other flows, such as boundary layers or flow in two-dimensional channels, show complete stability only at Reynolds Numbers below some critical value. Against the theoretical prediction of the stability of tube flow is the practical fact that the flow does become turbulent when sufficiently disturbed.

† Further discussion of reference 53 will be found in the paper by C. C. Lin entitled *Aspects of the Problem of Turbulent Motion* in this issue, p. 453.

‡ The investigation was supported by the USAF Office of Scientific Research.

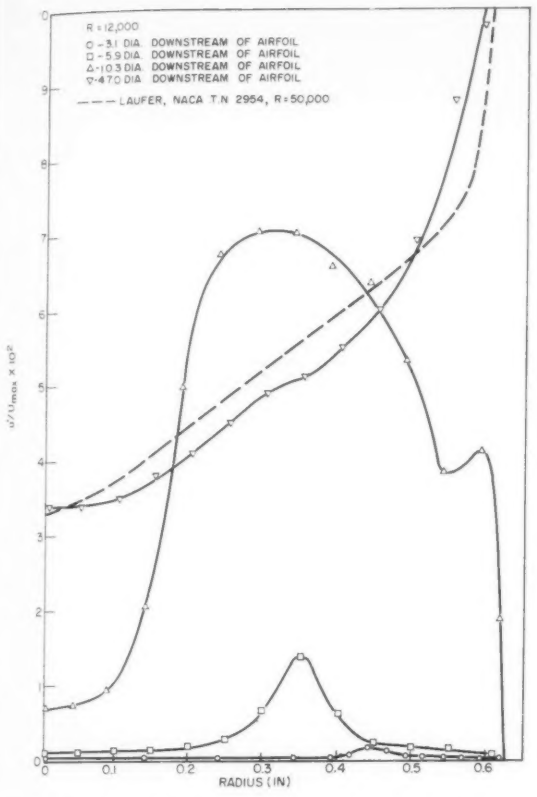
R = 12,000
○ - 3.1 DIA. DOWNSTREAM OF AIRFOIL
□ - 5.9 DIA. DOWNSTREAM OF AIRFOIL
△ - 10.3 DIA. DOWNSTREAM OF AIRFOIL
▽ - 47.0 DIA. DOWNSTREAM OF AIRFOIL
——— LAUFER, NACA T.N. 2954, R = 50,000
$u'/U_{max} \times 10^2$
RADIUS (IN.)

FIG. 1. Radial distribution of turbulence in wake of ring airfoil of radius 0.45 in.

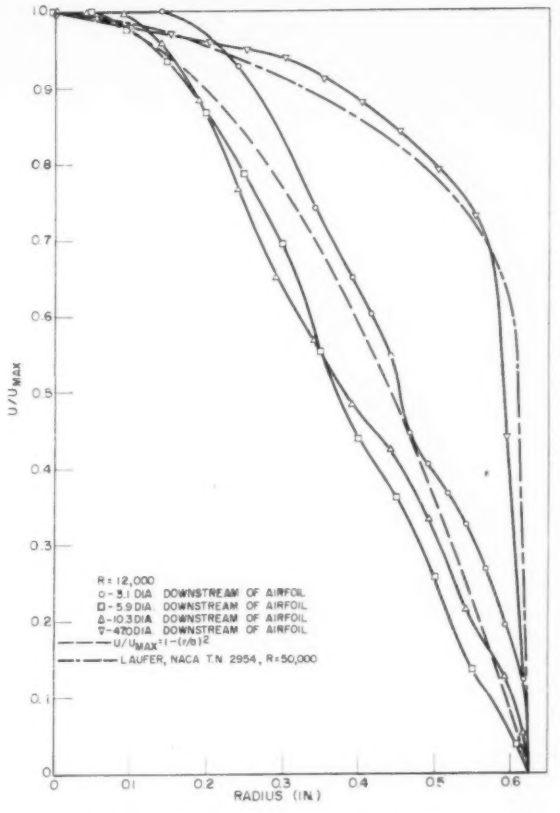
U/U_{MAX}
R = 12,000
○ - 3.1 DIA. DOWNSTREAM OF AIRFOIL
□ - 5.9 DIA. DOWNSTREAM OF AIRFOIL
△ - 10.3 DIA. DOWNSTREAM OF AIRFOIL
▽ - 47.0 DIA. DOWNSTREAM OF AIRFOIL
$U/U_{MAX} = 1-(r/b)^2$
——— LAUFER, NACA T.N. 2954, R = 50,000
RADIUS (IN.)

FIG. 2. Radial distribution of mean velocity downstream of ring airfoil.

Leite's measurements show that the flow is indeed stable up to the maximum Reynolds Number attainable in the tests (13,000) as long as the magnitude of the disturbance does not exceed a critical value, which depends on Reynolds Number. The measurements and their comparison with theory will be reported in detail elsewhere. Our concern here will be only with a few observations applicable to the study of transition.

The measurements which led to the conclusion that the flow is stable to small amplitude disturbances were made downstream of an oscillating sleeve 0.002 in. thick and 2 in. long, fitting closely to the inner wall of the tube. The measurements described here involved a larger amplitude disturbance, generated by a 0.9 in. diameter ring airfoil of about 0.1 in. chord, 0.003 in. thick, set at an angle of attack of about 6°. The ring was symmetrically placed inside the tube and was oscillated by means of an oscillator connected to the loud speaker coil.

Hot-wire measurements of the relative mean velocity U/U_{max} and relative root mean square fluctuation u'/U_{max} at a Reynolds Number of 12,000 are shown in Figs. 1 and 2. The fluctuation amplitudes were independent of frequency or amplitude of oscillation of the ring. The first measurements were made 3.1 diameters downstream of the ring. A weak turbulent wake with maximum fluctuations of about 0.2 per cent was measured at this station. Within the wake the fluctuations had the random character of turbulence. As we move downstream, the wake grows considerably in magnitude; however, the rate of spread is surprisingly small, the double angle being between 1.0 and 1.5 degrees. The fact that the tube flow is stable to small disturbances is probably responsible for the slow rate of spread of the wake. Between 10.3 and 47 diameters, the fluctuation profile changes radically to one very near that measured by Laufer[59] in fully developed turbulent flow in a tube.

The corresponding changes in the mean velocity profiles are shown in Fig. 2. Little change is noted between 3.1 and 10.3 diameters; in fact, at 10.3 diameters, the profile is slightly farther from the turbulent distribution than it is at 3.1 diameters. The fully developed parabolic laminar profile is shown for comparison. We note that, at 47 diameters, the mean velocity profile is near that measured by Laufer. The transition distance shown by these measurements seems rather long, though intermediate measurements between 10 and 47 diameters may show rapid adjustment at some intermediate position.

As was mentioned above, the ring was stationary at a Reynolds Number of 12,000 for the results shown in Figs. 1 and 2. In fact, no change in the fluctuation amplitude could be detected when the ring was oscillated. At a Reynolds Number of 8,000 the flow was stable when the ring was stationary, but transition re-

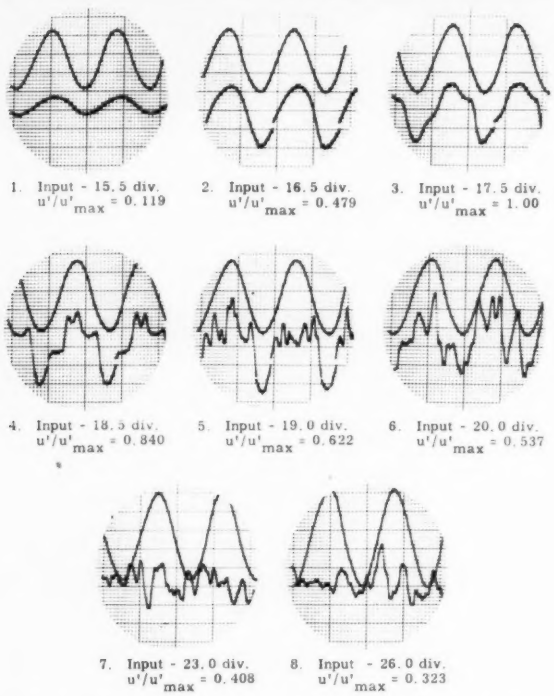
1. Input - 15.5 div. u'/u'max = 0.119
2. Input - 16.5 div. u'/u'max = 0.479
3. Input - 17.5 div. u'/u'max = 1.00
4. Input - 18.5 div. u'/u'max = 0.840
5. Input - 19.0 div. u'/u'max = 0.622
6. Input - 20.0 div. u'/u'max = 0.537
7. Input - 23.0 div. u'/u'max = 0.408
8. Input - 26.0 div. u'/u'max = 0.323

FIG. 3. Oscillograms of velocity disturbances for various imposed disturbances with ring airfoil. Upper trace is proportional to the current input to the disturbance generator; frequency 25 cycles per sec.; Reynolds Number 4,000. Lower trace shows hot-wire response at $r/a = 0.57$ (3.6 in.), 47 diameters downstream of generator. The amplification of the hot-wire response was decreased between exposures 2 and 3.

sulted when the ring was oscillated at any frequency or amplitude. At a Reynolds Number of 4,000, it was possible to oscillate the ring in the stable or unstable regimes as shown in Fig. 3, which represents photographs of the oscilloscope traces of hot-wire response for various input signals to the disturbance generator.

In Fig. 3, the amplitude of the upper traces on each exposure is proportional to the current being supplied to the disturbance generator. The lower trace is the response of a hot wire at $r/a = 0.568$, 47 diameters downstream of the disturbance generator. In exposure 1, the hot-wire response is very nearly a pure sine wave. Between exposures 1 and 2, the current to the generator has been increased only 6.5 per cent, while the hot-wire signal increased 400 per cent. In exposure 3, the hot-wire signal reaches a maximum ($u'/u'_{max} = 1$) and the wave is visibly distorted. It is noteworthy that, in exposures 4 and 5, the upper half of the wave is distorted much more than the lower; the velocity increases downward so that at this station, it is the low velocity half of the wave that is experiencing the maximum distortion. Characteristic turbulent fluctuations appear to have been achieved in exposures 7 and 8.

We may go a little further in describing the first stages of transition if we adopt a streamline picture similar to that found for the laminar boundary layer—that is, the disturbances in the tube would be represented by ring eddies alternating in the sign of their circulation.† Exposures 4 and 5 indicate that initially every second disturbance eddy breaks up. More observations will be required before the essential features of the breakup of the regular disturbances can be described closely.

(4) Turbulent Boundary Layer

In the previous section, a few features of the transition from laminar to turbulent flow in boundary layers and tubes were described. While transition can come about in many ways, the final result, the turbulent boundary layer, eventually develops certain characteristics independent of the way in which the laminar flow was destroyed. Sharp, distinct boundaries separate the turbulent flow in the layer from the potential flow in the free stream[56] and from the laminar flow upstream.[51] The outer boundary at any particular station wanders in and out and the upstream boundary of the turbulent layer wanders upstream and downstream in a random fashion. Gaussian distribution functions describe closely the positions of both boundaries.

The mean properties of the turbulent boundary layer and the effects of roughness and compressibility have been extensively investigated by the mixing length and similarity theories. In spite of the simplifications introduced, these theories have been remarkably successful in predicting effects of roughness and compressibility. Rather than discuss these theories, however, a more revealing approach is to consider the parameters which affect the mean velocity distribution. Many investigations[57–63] have demonstrated that the turbulent layer is divided into two distinct regions; an inner region adjacent to the wall in which the distribution is influenced by the viscosity, and an outer region adjacent to the potential flow in which the velocity defect at any position y/δ is governed by the friction velocity $\sqrt{\tau_0/\rho}$. These regions are separated by an "overlapping region" in which the laws governing the flow in the inner and outer regions overlap.

The three regions are described as follows. (1) The inner or laminar sublayer region is defined as the region within which the flow phenomena are governed by the friction at the wall—i.e., by Prantl's "law of the wall"[64]

$$u^* \equiv U/U_\tau = f(yU_\tau/\nu) \equiv f(y^*) \qquad (1)$$

where U is the mean velocity, U_τ is the friction velocity $\sqrt{\tau_0/\rho}$, y the distance from the wall, and ν the kinematic viscosity. Hama[65] gives a formula which approaches a linear relation $u^* = y^*$ for $y^* \to 1$ and fairs smoothly into the overlapping region described below at $y^* = 32$. (2) The "outer region," comprising the outer

† Theory and experiment indicate that, for a flat plate, the disturbances may be represented by eddies, with axes normal to the flow and with circulation of alterating sign. The eddies are carried downstream at a speed less than that of the main stream. Leite's measurements[51] indicate that this picture is somewhat idealized when applied to the disturbances in a tube, but qualitatively there appears to be some similarity with the configuration over a plate.

85 per cent of the layer at low speeds, is that portion of the boundary layer in which flow conditions are independent of viscosity, but dependent on the friction at the wall. The functional dependence in this region, first recognized by von Kármán,[66] is described by the functional equation

$$(U_1 - U)/U_\tau = g(y/\delta) \qquad (2)$$

where U_1 is the velocity at the edge of the boundary layer. This relationship describes the experiments for $y\,U_\tau/\delta^* U_1 > 0.045$, where δ^* is the displacement thickness of the boundary layer. (3) The region of overlap between the inner and outer regions was shown by Millikan[67] on the basis of functional reasoning, to have the semilogarithmic form. Clauser[57] gives the formulas

$$U/U_\tau = 5.6 \log (yU_\tau/\nu) + 4.9 \qquad (3a)$$

$$(U_1 - U)/U_\tau = -[5.6 \log(yU_\tau/\delta^* U_1) + 0.6] \qquad (3b)$$

which then describe the velocity profile between the points $y\,U_\tau/\nu = 32$ and $y\,U_\tau/\delta^* U_1 = 0.045$. Figs. 4 and 5 show the three regions defined by some of the experimental results.

While the three regions described above can be identified whether the turbulent flow be in a boundary layer, tube, or channel, there are important differences among the three, particularly in the outer region.† Deviations in the outer regions are ascribed to differences in the outer boundary condition.[68] The intermittency[56] of the turbulence in the outer region of the boundary layer has no counterpart in a tube or channel, since no outer potential flow exists there, and the resulting mean velocity distributions reflect this difference throughout the outer region. Over the inner region the distribution is independent of pressure gradient,[61] and over the outer region the distribution is independent of roughness if a small adjustment of the origin of y is introduced.[65]

† Fig. 4 shows that velocity distributions in the inner and overlap regions coincide closely for measurements in a tube (Laufer) and in a boundary layer (Kleanoff and Diehl). Laufer's measurements in a channel[60] are, however, higher than those shown, and no reason for the disagreement is evident. (See page 22 of reference 70.)

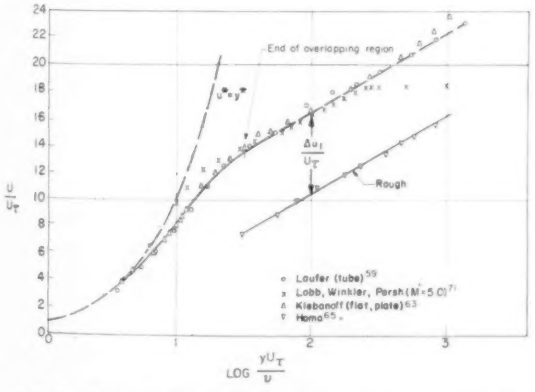

FIG. 4. Distributions of mean velocity near wall in turbulent flow. Effects of roughness and compressibility are shown. The Lobb *et al* data are plotted using values of ρ and μ at the wall.

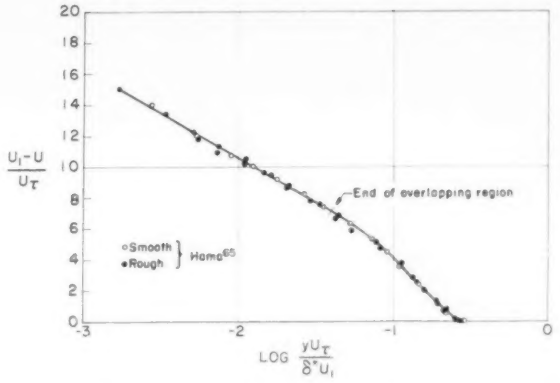

FIG. 5. Distributions of mean velocity in outer and overlapping regions of turbulent boundary layers on smooth and rough plates.

The effects of roughness are complicated by the fact that a quantitative measure of roughness depends not only on the height of the roughness element, but also on the distribution of the elements on the surface. However, as long as one deals with a particular type of roughness—e.g., sand—some interesting generalizations are possible. For instance, the influence of roughness is well represented by subtracting $\Delta u_1/U$ from the right-hand side of Eq. (3a).[57] The quantity $\Delta u_1/U$ is illustrated by some experimental results[65] plotted in Fig. 4. $\Delta u_1/U$ is a function of kU_τ/ν where k is the height of the roughness elements. The results of many investigations show that the effect of roughness on the skin-friction coefficient is essentially zero if $kU_\tau/\nu < 5$ (approx.)—i.e., if the roughness elements lie within the laminar sublayer, defined by $yU_\tau/\nu = y^* \cong 5$.[85]

The skin-friction coefficient law for smooth and rough plates follows from Eqs. (3a) and (3b) for the overlapping region. Adding the two equations results in the formula for local skin-friction coefficient,

$$\sqrt{2/c_f} = 5.6 \log (U_1\delta^*/\nu) + 4.3 - (\Delta u_1/U) \qquad (4)$$

One of the important results found empirically by Hama[65] is that the effect of roughness is not dependent on the outside flow and therefore the effects established for rough pipes can be applied to plates. This conclusion is true at least for "fully developed" roughness effect at zero pressure gradient.

Clauser[57] and Rotta[69] analyzed the turbulent boundary layer with pressure gradient and found conditions for "similar" velocity profiles in the outer region, under the influence of pressure gradient. They found the similarity parameter

$$G = \int_0^\infty \left(\frac{U_1 - U}{U_\tau}\right)^2 d\left(\frac{y\,U_\tau}{\delta^* U_1}\right) \qquad (5)$$

Clauser made extensive measurements and found $G = 6.1$ for constant pressure and $G = 10.1$ and 19.3 for two adverse pressure distributions empirically determined. The velocity profiles were similar in the sense that the above values of G held over the entire length of the plate and, when the results were plotted according to Eq. (2),

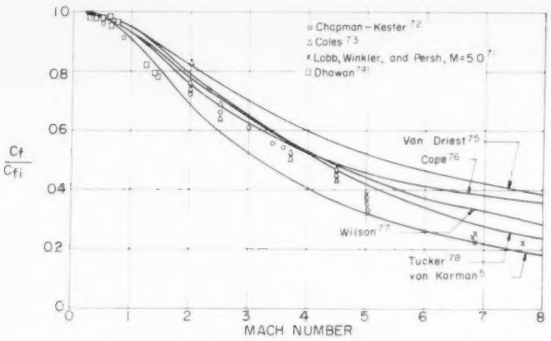

FIG. 6. Ratio of compressible to incompressible skin-friction coefficients for flat plate as function of Mach Number of external flow.

a separate relationship $(U_1 - U)/U_\tau$ vs. y/δ was defined for each pressure distribution. It was found further that the effect of pressure gradient could be represented by a $\Delta u_2/U_\tau$ analogous to the $\Delta u_1/U_\tau$ used in Eq. (4) to represent the effect of roughness. $\Delta u_2/U_\tau$ had different values for the two adverse pressure distributions for similar profiles referred to above.

A valuable review and analysis of the turbulent boundary layer in both compressible and incompressible flow was given by Coles.[70] The functional reasoning leading to the various relationships for the shearing stress and for the velocity distributions in the various regions of the boundary layer is given in detail.

Some of the details of the turbulent boundary layers at high Mach Numbers are determined by Lobb, Winkler and Persh[71] on the basis of total head and temperature transverses in the entrance section of a hypersonic tunnel. Free-stream Mach Numbers between 5 and 7.7 were achieved. The boundary layers were thick enough so that the measurements penetrated into the laminar sublayer. They show that, up to the highest free-stream Mach Numbers attained, the "law of the wall" $u^* = y^*$ (see Fig. 4) is valid, provided the properties μ and ρ are evaluated at the wall.

A remarkable feature of these results is the relatively large part of the boundary layer which conforms approximately with the law of the wall. The edge of the boundary layer is at log $y\,U_\tau/\nu = 2.4$, whereas in the low-speed boundary layer this point represents less than 15 per cent of the boundary-layer thickness. Further, results taken over a wide range of values of heat transfer from the wall to the stream show that the law of the wall is valid throughout. Outside of the inner or laminar sublayer region, the velocity profile closely satisfied a power law $U/U_1 = (y/\delta)^{1/n}$ where n decreased from 7 to 5.5 as the Mach Number increased from 5 to 7.7.

The effect of heat transfer on the skin-friction coefficient was found to be quite small. Further, calculations of the skin-friction coefficient by the use of Reynolds' analogy agreed to within 5 per cent with that given by the measured velocity gradient at the wall.

The first estimate of the effect of compressibility on the turbulent skin friction was made by von Kármán.[5] He simply introduced wall properties for an insulated plate as functions of the free stream Mach Number in his formula for skin friction at low speeds, and evaluated the skin-friction coefficient as a function of the free stream Mach Number. This method receives some a posteriori justification from the results of Lobb, Winkler and Persh,[71] described above, in that they find the law of the wall valid up to high Mach Numbers provided that values of density and viscosity prevailing at the wall are used in the calculation of u^* and y^*.

In the intervening years, virtually every variation of the mixing length and similarity theories has been used to calculate turbulent skin friction at high speeds. It is not possible to distinguish between their relative accuracy except by comparison with experiment. Fortunately, there have been many excellent measurements of the skin friction on flat plates and on cone-cylinder combinations. The determinations of friction coefficients have been made by the von Kármán momentum integral,[46] by overall drag measurements and by measurement of the force on a section of the plate suspended on a sensitive balance. Some of the available experimental results are plotted in Fig. 6 as the ratio of the skin-friction coefficient to that for incompressible flow vs. free-stream Mach Number. For comparison, those theoretical results which bracket the measurements are included. While there is some effect of Reynolds number on the ratio C_f/C_{f_i}, the major variation is with Mach Number. Fig. 6 indicates that the skin-friction coefficient on a flat plate as a function of Mach Number is slightly higher than von Kármán's estimate. A variation on von Kármán's method is given in reference 79, along with a comparison with experiment.

Investigations of separation of turbulent boundary layers for both incompressible and compressible flow have been directed toward finding the governing dimensionless parameters. In incompressible flow, a large body of experimental data over a wide range of Reynolds Numbers indicates that the velocity profiles satisfy the functional relation[80]

$$U/U_1 = [f(y/\theta), (\delta^*/\theta)]$$

where θ is the momentum thickness of the boundary layer. There is some disagreement between these measurements and the analyses of Clauser[57] and Rotta[69] in which the factor G [Eq. (5)] rather than δ^*/θ is proposed as a similarity parameter. The factors G and δ^*/θ are connected by the relation

$$\delta^*/\theta = 1/(1 - G\sqrt{C_f/2}) \qquad (6)$$

However, Clauser found that the boundary layer was quite sensitive to small lateral pressure gradients, so that his measurements, in which precautions were taken to eliminate the lateral gradients and which showed good correlation with G rather than with δ^*/θ, might conceivably satisfy a different similarity parameter than those of Schubauer and Klebanoff.[80] Von Doenhoff and Tetervin[81] give a semiempirical analysis for the growth of the turbulent layer in an adverse pressure

gradient. Their measurements show that separation occurs for a value of δ^*/θ between 2.6 and 2.8. Chapter 22 of reference 1 gives a more complete account of these and other investigations.

Separation of supersonic turbulent boundary layers has been studied by means of shock-wave interactions. In references 15 and 16 incident shocks were not strong enough to cause separation of the turbulent layer. In reference 17, when collars with step leading edges were fitted around the body of revolution, $\Delta p/q_1$ for flow separation varied with $Re^{-1/5}$. Bogdonoff[82] gives a summary of an extensive series of tests carried out at Princeton University. Optical studies and wall pressure distributions were made with incident shocks on a flat plate, with steps and with normal shocks in a diverging channel. Separation points were located with a total head tube. It was found, for instance, that separation occurred when a 10° shock intersected the boundary layer at $M = 2.9$ and 3.8; shocks of lesser strength did not cause separation. Reynolds Numbers based on momentum thickness of several thousand were used.

Incompressible flow about a yawed cylinder may be analyzed by the independent consideration of one of the equations of motion, provided the boundary layer is laminar (Section 2). For a turbulent layer, the fact that spanwise gradients of the velocity exist—by reason of the turbulent fluctuating velocities—nullifies the independence of the equations. However, the experimental evidence[53, 28] is mixed as to whether a practical degree of independence still exists. Careful investigations on yawed flat plates[83] indicate that the boundary-layer properties are not the same functions of the normal distance from the leading edge as they would be if the independence principle were valid. On the other hand, measurements on yawed airfoils[84] indicate approximate validity of the principle.

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Aspects of the Problem of Turbulent Motion*

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(I) Introduction

In 1934, Theodore von Kármán[1] delivered a general lecture on turbulent flow at the International Congress of Applied Mechanics. The next year, Taylor[2] published his statistical theory of turbulence. General systematic development of the theory was then made by von Kármán[3] and jointly with Howarth.[4] Since that time, there has been an almost steady progress in our knowledge of the mechanics of turbulent motion, both theoretical and experimental, although the basic problems still remain largely unsolved. Mathematical aspects of the analysis of turbulent motion have also received considerable attention.

In this short article, it is impossible to describe all phases of our present knowledge of turbulent motion to any reasonable degree of completeness, or even to sketch them briefly. The selection of material is therefore partly determined by the writer's personal interest and competence. On the whole, emphasis has been placed on the physical mechanism of turbulent motion. For example, on the experimental side, the discussion of such practically useful data on coefficients of heat transfer and skin friction at supersonic speeds will be omitted, but some space will be devoted to the description of such topics as the intermittent nature of turbulent flow. It is hoped that the former items will be covered more fully in this issue in the articles on boundary layer and heat transfer. Description of experimental technique will be entirely omitted as beyond the scope of this survey and the competence of the writer. On the theoretical side, the survey will be largely devoted to the statistical theory of turbulence, along the lines of Taylor and von Kármán, as this appears to be the direction where successful future developments will be made. Omitted then are the approaches such as those used by Burgers[5] and by Hopf[6] and the discussion of the mathematical foundation such as that made by Kampé de Fériet.[7] The theories of turbulent shear flow, such as the mixture length theory, will be discussed only briefly (cf. Section IX).

(II) General Description of Turbulent Flow

Before we go into the details of the various phases of turbulent motion, we shall give first a qualitative discussion of several of its outstanding features—namely,

(1) randomness of turbulent fluctuations
(2) diffusive nature of turbulent motion
(3) rapid dissipation of mechanical energy into heat
(4) intermittency of small-scale fluctuations

It is clear that one cannot regard any fluctuating flow as turbulent motion in the strict sense; otherwise it would be senseless to try, for example, to find quantitative laws for the mean velocity distribution of turbulent flow through a pipe. The fact that empirical laws can be found for such distributions suggests that a definite statistical state has been reached analogous to the random motion of molecules in a gas. Obviously, in actual experiments, there may be regular oscillations or cellular motions on top of turbulent fluctuations. The precise separation of regular oscillations from turbulent motions may be difficult—e.g., in the problems of atmospheric turbulence; but in most cases a general agreement can be reached.

The random nature of turbulent fluctuation naturally leads to the use of statistical treatment. The introduction of statistical correlation and spectrum of turbulence is then a logical step. Significant developments of the statistical theory, however, were made only after the measurement of such quantities became possible through the improvement of the hot-wire technique. Randomness also leads to the thought that the frequency distribution of the velocity fluctuations is Gaussian (cf. Fig. 1). The extent to which such a hypothesis holds as an accurate approximation will be discussed in Section IV.

The diffusive nature of turbulent flow has long been known. Associated with this is the transfer of fluid properties—such as temperature, material, and momentum—whenever there is a gradient of these quantities in the mean flow. Such problems are naturally of great practical importance. Although much effort

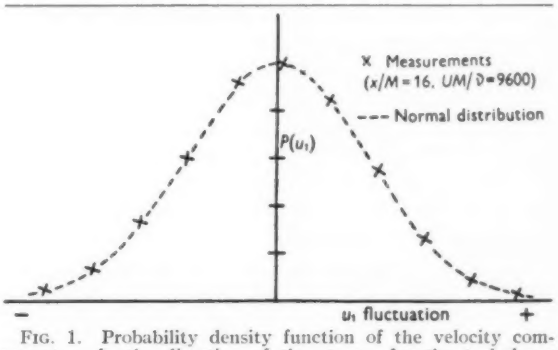

Fig. 1. Probability density function of the velocity component u_1 in the direction of the stream for the turbulence generated by a square-mesh grid in a wind tunnel (after Townsend[72]).

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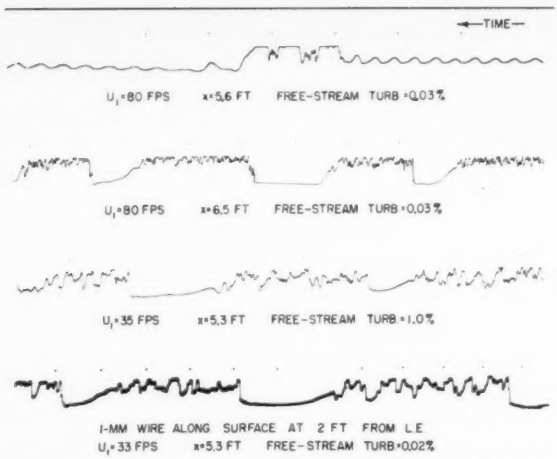
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1-MM WIRE ALONG SURFACE AT 2 FT FROM L.E.
U₁=33 FPS x=5.3 FT FREE-STREAM TURB=0.02%

FIG. 2. Oscillograms of u-fluctuation in transition regions in a boundary layer over a flat plate obtained with hot-wire probe 0.013 in. from surface. Time interval between dots, 1/60 sec. (after Schubauer and Klebanoff[10]). Free-stream speed is denoted by U_1, distance from leading edge by x.

has been devoted to the theory of turbulent diffusion and transfer, most of our theoretical knowledge remains semiempirical.

The rapid dissipation of mechanical energy of turbulent flow is related to its diffusive nature. Taylor[8] pointed out that the stretching of vortex lines in turbulent motion transfers the kinetic energy from large eddies into small eddies and that the energy contained in the small eddies are then rapidly dissipated into heat. The predominant part of the turbulent energy is usually contained in the large eddies (at high Reynolds Numbers of turbulence), but the predominant part of the dissipation is attributable to the small eddies which contain a negligible part of the kinetic energy.[9] Such a physical picture is the basis for Kolmogoroff's theory (Section VIII). Each of the steps in the above process can be associated with certain terms in the Navier-Stokes equations. The nonlinear terms and the three-dimensional nature of the motion are essential for the mechanism of stretching. The effects of the viscous terms, having derivatives of the highest order, become accentuated as the scale of the motion becomes small. It seems that any reasonable mathematical model of turbulence should exhibit such basic features.

The intermittent nature of turbulent flow has been noted in so many instances that it must be regarded as one of its essential features. It is more prominent in connection with the transition phenomena[10] (Fig. 2) and with the border region of a turbulent boundary layer,[11] a turbulent jet,[12] or a turbulent wake.[13] In fact, even in the earliest transition experiments in a pipe, it has been noted that when the pressure drop is increased due to transition to turbulence, the additional drop is not distributed uniformly along the pipe, but turbulent mixture and laminar movement alternated more or less regularly.† A hot wire placed at a given station would then be expected to show a record of the type found by Schubauer and Klebanoff in turbulent boundary layers.[10] The "spotty" nature of the turbulent region in boundary-layer transition was first pointed out by Emmons.[14]

Even in fully developed turbulence, it has been shown by Batchelor and Townsend[15] that there is intermittency in the spacial structure. Also, the theory of stability of parallel motions indicates a spatial amplitude distribution of the intermittent nature for the damped oscillations.

This intermittent nature of turbulent flow has a profound theoretical interest. From the basic Navier-Stokes equations, it is clear, mathematically, that the viscous term is negligible at high Reynolds Numbers unless the scale of the motion becomes correspondingly small. Thus, there are two types of limiting behavior possible. It is interesting to note that both types coexist in a turbulent flow field. Of course, in a region away from the boundary, unless this is the case, either the motion would be essentially that of a perfect fluid or the turbulent motion would not have a finite scale as the Reynolds Number becomes infinite. The spotty spatial nature of the turbulent motion at large Reynolds Numbers is then a necessary consequence of the finiteness of the scale.††

(III) Transition from Laminar Flow to Turbulence

When a laminar flow at sufficiently high Reynolds Numbers is disturbed, the motion generally becomes turbulent. The question then arises whether the disturbance must be sufficiently strong to induce transition or whether the flow is inheritantly unstable with respect to infinitesimal disturbances which will grow as regular oscillations, leading eventually to the transition phenomenon. By considering the disturbances on the boundary layer caused by the residual turbulence in the free stream, Taylor[16] was able to give a definite correlation of the observed transition Reynolds Number with the intensity and the scale of free-stream turbulence when the intensity is 0.2 per cent or higher. This shows that many transition phenomena are indeed dominated by external disturbances and depend only secondarily, if at all, on the inheritant instability of the boundary layer with respect to infinitesimal disturbances. On the other hand, when the turbulence level is low, regular oscillations have indeed been observed by Schubauer and Skramstad[17] at low speeds and by Laufer[18] at supersonic speeds. The agreement between the observed characteristics with the theory of small oscillations leaves no room for doubt that these oscillations do play an important role in the phenomenon of transition.

The study of such small disturbances in a laminar flow —or the theory of hydrodynamic stability, as it is usually

† See discussions of Reynolds' observations in Prandtl's article, Durant's *Aerodynamic Theory*, Vol. 3, p. 121.

†† For more detailed studies of the free turbulent boundary, see Corrsin, S., and Kistler, A. L., NACA TN 3133, 1954; also Phillips, O. M., Proc. Camb. Phil. Soc., Vol. 51, pp. 220–229, 1955.

called—has an enormous literature. An account of the present status of our knowledge is given in a recent monograph[19] and will not be attempted here. Suffice it to mention that instability of laminar motion falls into two classes: one leading to another state of laminar motion (convection cells, cellular vortices between rotating cylinders), another leading to turbulence (instability of channel flow, boundary layer over a flat plate). In the first class, the physical mechanism causing the instability is usually fairly obvious, and viscous forces exert a stabilizing influence. In the second class, viscous forces actually play an essential role in causing instability by building up the necessary Reynolds stress which transfers the energy from the basic flow to feed the disturbances.

At the present time, only limited information can be obtained about the actual mechanism of transition from the study of these small oscillations. Much work remains to be done. Extensive calculations of the amplification of these oscillations appear to be a useful step.

On the other hand, there are direct experimental observations of the transition phenomenon. The water-table experiment of Emmons[14] showed isolated patches of turbulence which suggested to him the theory of transition by formation and growth of turbulent spots. Briefly, in a region of relatively large Reynolds Numbers where the boundary layer is unstable, turbulence spots are formed more or less at random, presumably due to external disturbances and oscillations in the boundary layer. These spots, which have very much smaller scales than that of regular laminar oscillations, seem to form characteristic entities themselves and are propagated downstream in a characteristic manner. Far downstream, there is a whole agglomeration of these spots, and the fully developed turbulence is achieved. In the transition zone, with a hot wire located at a given point, one would therefore expect an intermittent turbulent-laminar record.

Recently, Schubauer and Klebanoff[10] made hot wire observations in the transition region in a boundary layer and confirmed the ideas proposed by Emmons. The propagation and growth of these turbulent spots is found to be essentially identical with that described by Mitchener.[20]

The hot-wire observations are of course able to yield more detailed information. One very interesting phenomenon is the "calming" of the flow following the passage of a turbulent spot. At a point in the laminar boundary layer when regular oscillations are present, the artificial stimulation of transition causes the oscillations to be temporarily suppressed after the passage of the turbulent fluid. For other details, the reader is referred to reference 10. But it is already clear that the basic character of transition makes it impossible to define a clear-cut transition point.

Nevertheless, from the engineering standpoint, one would like to be able to establish a conventional definition of a transition point based on a particular method of observation. Indeed, one would like to be able to correlate various experimental results by using properly chosen parameters. To this end, the instability theory is often found useful in suggesting the proper parameters to be used and in predicting the general trend of the influence of various factors, such as the pressure gradient along the boundary, surface temperature, etc.

The practicability of making such correlation of experimental data is exemplified by the above-mentioned study of Taylor on the effect of intensity and scale of turbulence on the transition point. Such correlations are not easy when there are a number of factors influencing transition. For example, in recent experiments on the transition to turbulence of supersonic boundary layers, many of the results reported in the literature were apparently inconsistent with each other. An attempt at a systematic survey and analysis of existing data was made by Probstein and the present writer.[21]

The effect of surface roughness on transition has received considerable attention. A review of the earlier literature was made by Dryden.[22] Additional information were found by Schubauer and Klebanoff.[23] At high supersonic speeds, it has been suggested that the effect of roughness may be less important because the disturbance produced is expected to propagate outside of the boundary layer as compression waves.[24] There seems to be experimental support to this idea.

The effect of surface temperature on transition has received wide interest, especially at supersonic speeds. This is largely suggested by the instability theory.[25] The general trend of such predictions is borne out in actual transition studies.[21]

(IV) Statistical Theory of Turbulence

We shall now turn to the discussion of fully developed turbulent flow. It has long been recognized that the details of turbulent motion is so complicated that statistical considerations must be used for its description. One of the earliest of such applications is the concept of the Reynolds stress, which gives a very illuminating general description of the basic mechanism of turbulent shear flow. However, up to the present time, very little further progress has been made in the basic theory of shear motion because of its inherently complicated nature. Most of the progress has been made in the much simpler problem of homogeneous turbulence without mean motion. Superficially, one might think that there is little to be known about such fluid motions. Actually, the very absence of mean motion allows one to go more deeply into the inherent nature of the turbulent motion itself. Many basic concepts have been developed in the study of homogeneous turbulence, and these concepts now gradually find their way into the study of shear flow.

One of the basic concepts in the discussion of statistical averages is the frequency of occurrence, or the distribution function. We consider a function $F(u, v, w; x, y, z, t)$ which gives for each point $P(x, y, z)$ and each

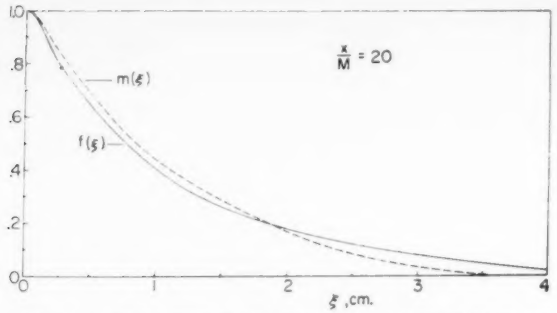

FIG. 3. Double correlation functions involving velocity and temperature fluctuations u and ϑ behind a heated grid set perpendicular to a uniform air stream.

$$m(\xi) = \frac{\overline{\vartheta(x)\vartheta(x+\xi)}}{\vartheta'(x)\vartheta'(x+\xi)}$$

$$f(\xi) = \frac{\overline{u(x)u(x+\xi)}}{u'(x)u'(x+\xi)}$$

where x and ξ denote distance in the direction of the stream and a dash indicates the root-mean square value. M is the mesh width of the grid. (After Kistler, O'Brien, and Coorsin[23]).

instant t the probability that the turbulent velocity shall lie in the range $(u, u + du; v, v + dv; w, w + dw)$, or briefly $(u_i, u_i + du_i)$. If this function is known, then the Reynolds shear is given by a formula of the kind

$$-\rho\overline{uv} = -\rho\int_{-\infty}^{\infty}\iiint F(u,v,w;\ x,y,z,t)uv\,du\,dv\,dw \quad (4.1)$$

for each point $P(x, y, z)$ at each instant t.

To analyze the structure of turbulence, one needs to know also the joint probability distribution at several points. For example, if we are interested in the correlation of velocity at two points P' and P'', then we must know a distribution function of the form $F(u_i', x_i'; u_i'', x_i''; t)$. The correlation of the x-components of the velocities at these points is then given by

$$\overline{u(P')\,u(P'')} = \iint F(u_i', x_i';\ u_i'', x_i'', t)\,u'u''\,d\tau(u_i')d\tau(u_i'') \quad (4.2)$$

Further generalizations of joint probability distributions involve more than two points and quantities more general than velocity fluctuations (cf. Fig. 3, 4).

Some experimental information is available regarding the distribution function $F(u_i, x_i, t)$. The Gaussian distribution has been found, in many cases, to be a fairly good representation for each component.[26, 27] Some skewness is of course to be expected in any anisotropic case. In the isotropic case, where the statistical properties of the motion are essentially independent of direction, the Maxwellian distribution of velocity holds approximately.

There is also some indication that the joint probability distribution at two points is approximately jointly Gaussian. However, this cannot be accurate because the triple correlation $\overline{u^2(P)u'(P')}$ for two points P, P' lying on a line parallel to the x-axis is known to play an important role in the mechanics of turbulent motion. This quantity would be identically zero for a truly Gaussian distribution. The simplest way to get a measure of the accuracy of the Gaussian hypothesis is to measure the moments of the velocity difference at such two points: $u(P') - u(P)$. Such measurements have been made by R. W. Stewart.[28] The flatness factor is found to be close to three except for small distances (Fig. 5).

(V) General Equations of Homogeneous Turbulence

The simplest turbulent flow is the idealized case of an infinite field of turbulence which is devoid of mean motion and is statistically *homogeneous*, or even *isotropic*. Homogeneity means that the statistical properties of the field are independent of the particular position in the field, and isotropy means that they are independent of direction. Clearly, one of the most important quantities in such a field is the intensity of turbulence. To get an idea of the structure of the field, it is natural to consider quantities like the correlation tensor

$$\overline{u_i(P)\,u_j(P')} = R_{ij}(\xi_k) \quad (5.1)$$

where $u_i(P)$ and $u_j(P')$ are the velocity components at two points P and P', and ξ_k are the components of the vector PP'. Because of homogeneity, the correlation tensor R_{ij} is a function of the vector $\{\xi_k\}$. The condition of incompressibility then yields the relation

$$\partial R_{ij}/\partial\xi_j = 0 \quad (5.2)$$

In the isotropic case, von Kármán[3] has shown that R_{ij} is of the form†

$$R_{ij} = \overline{u^2}\{[f(r) - g(r)]\,(\xi_i\xi_j/r^2) + g(r)\,\delta_{ij}\} \quad (5.3)$$

and that Eq. (5.2) leads to

$$g = f + (r/2)\,(\partial f/\partial r) \quad (5.4)$$

† Notice that $\overline{u^2}f = R_{11}$, $\overline{u^2}g = R_{22}$ for $(\xi_1, \xi_2, \xi_3) = (r, 0, 0)$.

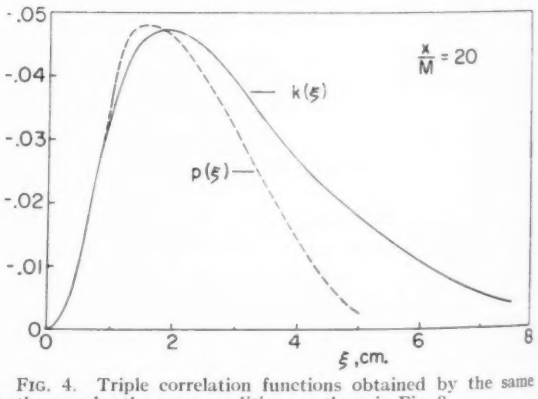

FIG. 4. Triple correlation functions obtained by the same authors under the same conditions as those in Fig. 3.

$$k(\xi) = \frac{\overline{u^2(x)u(x+\xi)}}{u'^2(x)u'(x+\xi)}$$

$$p(\xi) = \frac{\overline{u(x)\vartheta(x)\vartheta(x+\xi)}}{u'(x)\vartheta'(x)\vartheta'(x+\xi)}$$

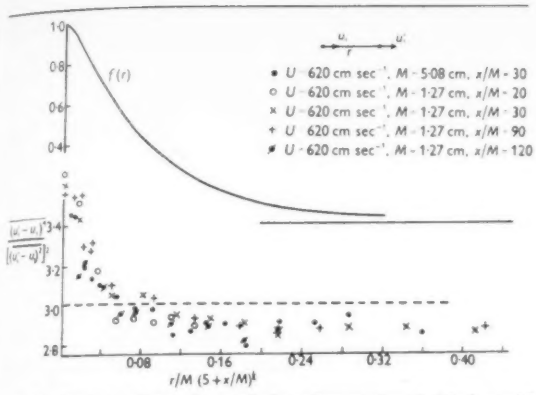
f(r)
• U - 620 cm sec⁻¹, M - 5·08 cm, x/M - 30
○ U - 620 cm sec⁻¹, M - 1·27 cm, x/M - 20
× U - 620 cm sec⁻¹, M - 1·27 cm, x/M - 30
+ U 620 cm sec⁻¹, M - 1·27 cm, x/M - 90
✱ U - 620 cm sec⁻¹, M - 1·27 cm, x/M - 120
r/M (5 + x/M)^½

FIG. 5. Flatness factor for turbulent fluctuations behind a grid across a uniform air stream (after Stewart[28]).

Thus, all velocity correlations of the second order can be expressed in terms of only one correlation function. Von Kármán and Howarth[4] have shown that the same is true of correlations of the third order associated with two points P and P'. They have further shown that the correlation between pressure and velocity fluctuations vanishes identically. With these preliminary kinematical results, they considered the dynamics of isotropic turbulence and derived the well-known Kármán-Howarth equation

$$\frac{\partial}{\partial t}(\overline{u^2}f) + (\overline{u^2})^{3/2} r^{-4} \frac{\partial}{\partial r}(r^4 h) = 2\nu\overline{u^2}\, r^{-4} \frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r}\right) \tag{5.5}$$

from the Navier-Stokes equations.

It is often physically convenient to think in terms of the spectral theory. The Fourier transform of the correlation tensor gives the spectral tensor

$$\Phi_{ij}(\kappa_l) = \frac{1}{8\pi^3} \iiint R_{ij}(\xi_k) e^{-i\kappa_m \xi_m}\, d\tau(\xi_k) \tag{5.6}$$

in terms of which we may define

$$E(\kappa) = \frac{1}{2} \iint \Phi_{ii}(\kappa n_l)\, dS(n_l) \tag{5.7}$$

which gives the distribution of the energy of turbulent motion among the various scales.

The above relations for correlation functions naturally have their counterparts in the spectral theory. The relation Eq. (5.2) corresponds to

$$\Phi_{ij}\kappa_j = 0 \tag{5.8}$$

From this relation, Kampé de Fériet[29] has shown that Φ_{ij} is of the form

$$\Phi_{ij} = A(\kappa)[\delta_{ij} - (\kappa_i\kappa_j/\kappa^2)] + B(\kappa)\varphi_i\varphi_j^* \tag{5.9}$$

where $A(\kappa)$ and $B(\kappa)$ are functions of κ, $\{\varphi_i\}$ is a unit vector, and $\{\varphi_i^*\}$ is its complex conjugate. In the isotropic case, $B = 0$.

The one-dimensional spectrum $F(\kappa)$, defined by the Fourier relations

$$\left.\begin{aligned} F(\kappa) &= \frac{2\overline{u^2}}{\pi}\int_0^\infty f(r)\cos\kappa r dr \\ \overline{u^2}f(r) &= \int_0^\infty F(\kappa)\cos\kappa r d\kappa \end{aligned}\right\} \tag{5.10}$$

is often found convenient from the experimental point of view.[30] (Cf. Fig. 6). It is easily verified that

$$E(\kappa) = (1/2)[\kappa^2 F''(\kappa) - \kappa F'(\kappa)] \tag{5.11}$$

The cosine transform of the Kármán-Howarth equation gives a relation for $F(\kappa)$, and this can be rewritten in terms of $E(\kappa)$ in the form

$$(\partial E/\partial t) + W(\kappa, t) = -2\nu\kappa^2 E \tag{5.12}$$

where $W(\kappa, t)$ is related to the triple correlation function.

The Eq. (5.5) of Kármán-Howarth [or the equivalent spectral Eq. (5.12)] is a relation giving the time rate of change of the double correlation function, but it is seen to contain the correlation function of the third order. If we consider the differential equation for the change of the latter function,† we shall find that correlations of still higher orders appear. The nonlinear terms in the Navier-Stokes equations prevents us from obtaining a closed system.

Further developments of the general theory thus appear to be impossible without introducing specific assumptions. Still, the above results serve as a framework in which the experimental data may be analyzed. There are many deductions, which, together with the relevent experimental data, serve to enhance our understanding of the mechanics of turbulent flow.

† N. Hu appears to be the first one to carry out such a derivation in an unpublished paper.

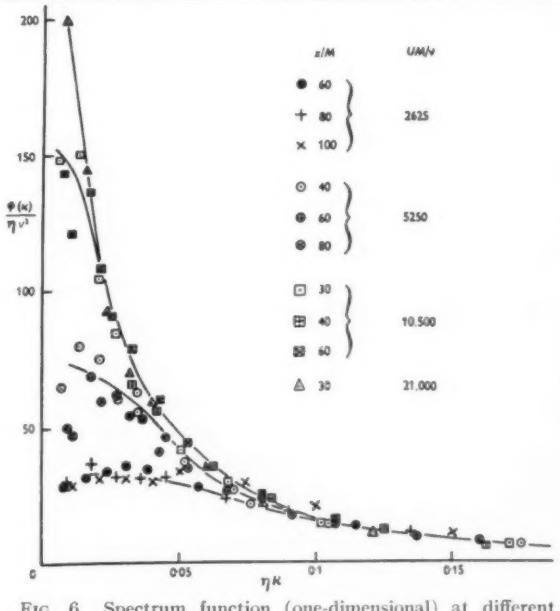

FIG. 6. Spectrum function (one-dimensional) at different decay times and different Reynolds Numbers (from Stewart and Townsend[45]). $\varphi(\kappa)$ is $\frac{1}{2}$ of $F(\kappa)$ defined by Eq. (5.10), and η is identical with l defined by Eq. (7.2).

The above theoretical developments for isotropic turbulence can also be carried out for the anisotropic case, but the essential difficulties remain the same.

(VI) Approximate Theories

In order to complete the theoretical formulation, certain approximations are needed. Obviously, if the terms in the triple correlation is negligible in Eq. (5.9), we have a well-defined mathematical problem. This is the case of low Reynolds Numbers considered by von Kármán and Howarth,[4] who gave solutions of the self-preserving type in terms of confluent hypergeometric functions. Further studies of this case were made by Batchelor and Towsend.[31, 32]

It should be pointed out that in this approximation the nonlinear terms in the Navier-Stokes equations are neglected, and the treatment can be handled in terms of these equations themselves. Such an approach has been made by Reissner[33] and studied in greater detail by Batchelor.[32] It should also be borne in mind that Taylor[8] has pointed out that one of the most important processes of turbulent motion is expressed by the term involving the triple correlations. Thus, the treatment of this case is of interest only to the final stage of a decaying turbulence and does not particularly contribute to the understanding of the physical processes of turbulent diffusion.

Approximate theories involving the essential triple correlations have been developed in two different ways. The first is based on the assumption of a relation between the triple and double correlation functions, or rather a relation between the transfer function and the spectrum in the spectral theory, Eq. (5.12). Such assumptions were proposed by Obukhoff,[34] Heisenberg,[35] von Kármán,[36] and Kovasznay.[37] Von Kármán's formula is

$$\int_0^\kappa W d\kappa = C \left\{ \int_\kappa^\infty [F(\kappa)]^\alpha \kappa \, d\kappa \right\} \times \left\{ \int_0^\kappa [F(\kappa)]^{-(3/2)-\alpha} \kappa^{(1/2)-\beta} \, d\kappa \right\} \quad (6.1)$$

Heisenberg's formula is obtained if we put $\alpha = (1/2)$, $\beta = -(3/2)$. The latter is the most widely used. It is obtained by considering the effect of the smaller eddies as having a dissipative action on the large eddies with an equivalent kinematic viscosity

$$\nu' = C \int_\kappa^\infty \{ F(\kappa)/\kappa^3 \}^{(1/2)} d\kappa \quad (6.2)$$

A second approximate theory was developed recently by Proudman and Reid[38] using the equations for the change of both the double correlation and the triple correlation. In the latter equation, the terms in quadruple correlations are expressed in terms of double correlations by the quasi-Gaussian approximation. This means that correlations of the fourth order are assumed to be connected with those of the second order as if the joint probability distributions were Gaussian. The triple correlations, however, do not vanish, as would be the case in the true Gaussian distribution.

Neither one of the approximate theories has as yet been fully exploited. Calculations of the correlation spectrum, based on Heisenberg's formula, have been made by Chandrasekhar[39] and compared with experiments by Proudman.[40] There seems to be general agreement, although there is some difficulty about the value of the constant factor appearing in the formula.

(VII) Self-Preserving Hypotheses

The development of the explicit theories discussed above has been preceded by the introduction of self-preserving hypotheses. These hypotheses give definite relations for the law of decay and change of scale, although their introduction still leaves more unknowns than equations. Since these hypotheses are often used as additional assumptions in the framework of the above theories and since they also give results by themselves independent of such theories, they deserve a separate treatment.

Self-preserving hypotheses were first introduced by von Kármán and Howarth; a survey of various hypotheses was made by von Kármán and Lin.[41] Here we shall mention a particular one[42] as an example, since it has considerable theoretical merit and has also received experimental support.

We assume that there is self-preservation of the correlation function except for the largest eddies. Mathematically, this means that we assume

$$\left.\begin{aligned} \overline{u^2}[1 - f(r,t)] &= v^2 F(r/l) \\ \overline{u^3}\, h(r,t) &= v^3 H(r/l) \end{aligned}\right\} \quad (7.1)$$

where $u = (\overline{u^2})^{1/2}$, and v and l are typical velocity and length scales dependent on time alone. It then follows from the Kármán-Howarth equation that

$$v = (\nu\epsilon)^{1/4}, \qquad l = (\nu^3/\epsilon)^{1/4} \quad (7.2)$$

where ϵ is the rate of energy dissipation. The law of decay is

$$\overline{u^2} = \alpha t^{-1} + \beta \quad (7.3)$$

This law of decay has been shown to be in good agreement with experimental results. Extension of the above ideas has been made by Goldstein,[43] who also suggested a two-grid experiment to distinguish the above law of decay (and other similar laws) from the special law

$$\overline{u^2} = \alpha t^{-1} \quad (7.4)$$

obtained in many single-grid experiments. Such two-grid experiments have been made recently by Tsuji and Hama[44] indicating the necessity of the inclusion of the additive constant (Figs. 7, 8). A comparative study of the various hypotheses along this line were made by the present writer,[45] in which the experimental work of Stewart and Townsend[46] is discussed.

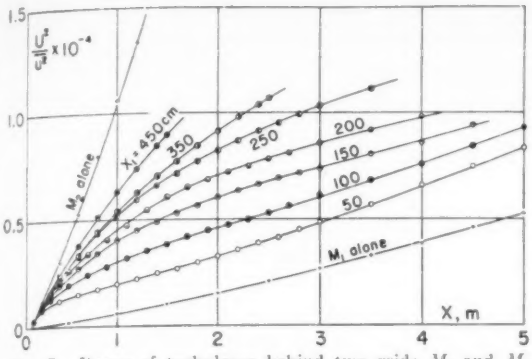

FIG. 7. Decay of turbulence behind two grids M_1 and M_2 with mesh widths 5 cm. and 1 cm., respectively (M_2 downstream from M_1). x is the distance behind the second grid M_2, x_1 is the distance between the grids, U is the mean velocity, and $\overline{u^2}$ is the mean-square turbulence. (After Tsuji and Hama[44].) Data show departure from Eq. (7.4).

(VIII) Large Eddies and Small Eddies

Taylor[9] has shown that, with a reasonably large Reynolds Number of turbulence, energy of the turbulent motion is contained in the larger eddies while nearly all the energy dissipation is associated with the smaller eddies contributing a negligible fraction of the energy. He also advanced the idea that the stretching of the vortex lines in a diffusive turbulent field is the mechanism for supplying energy to the smaller eddies. Later authors describe such a process as a cascade process of energy transfer down the scale of eddies.

The above picture leads to the plausible assumption that the smaller dissipative eddies are isotropic, that their behavior depends only on the quantities ϵ and ν, and that consequently there is a definite spectrum based on the velocity and length scales v and $\eta = l$, given by Eq. (7.2). However, such assumptions are justified only in the case of a virtually complete separation of energy-containing large eddies and the dissipative small eddies—i.e., if the general scale of turbulence L is much larger than the scale η. The transfer of energy from the large eddies to the small eddies is proportional to $\rho u^3/L$, which must be proprotional to ϵ. The condition $L \gg \eta$ then leads to the conclusion that the Reynolds Number of turbulence uL/ν must be very large.

Such is the general basis of the theory of locally isotropic turbulence proposed by Kolmogoroff,[47] Obukhoff,[34] and, later, apparently independently by Onsager,[48] Heisenberg,[35] and von Weizsäcker.[49] In the spectral formulation, Kolmogoroff's theory leads to the energy spectrum in the form

$$E = v^2\eta\, e\,(\kappa\eta) \tag{8.1}$$

for sufficiently large values of κ. For very high Reynolds Numbers, one assumes that there is a significant lower subrange of the spectrum, Eq. (8.1), which is independent of ν. We then have

$$E \sim \epsilon^{2/3}\kappa^{-(5/3)} \tag{8.2}$$

a result first obtained by Obukhoff.[34] The results discussed here are independent of the behavior of the larger eddies. They are, therefore, applicable to shear flows as well provided only smaller eddies are considered (hence the name *local isotropy*). There is some experimental evidence for Kolmogoroff's theory in shear flows.[50, 51]

Concerning the large eddies, the Loitsiansky parameter should be mentioned. If one assumes that the correlation functions $f(r,t)$ and $h(r,t)$ vanish sufficiently rapidly at infinity, then it is easy to show that the dimensional parameter

$$\Lambda = u^2 \int_0^\infty fr^4 dr \tag{8.3}$$

remains constant. This was pointed out by Loitsiansky.[52] The present writer has shown that in the spectral formulation, under the same assumptions for the behavior of the correlation functions, the spectrum is given by

$$E(\kappa) = (1/3\pi)\Lambda\kappa^4 + \ldots \tag{8.4}$$

and that the invariance of Loitsiansky's parameter is an indication of the permanence of the large eddies.

Such theories have been developed to include general homogeneous turbulence by Batchelor.[32] Recently, questions have been raised against the validity of the assumptions regarding the behavior of such correlations functions at infinity.[53, 38] At the present time, the existence of a time-independent parameter seems to be in doubt. This of course influences a number of developments based on the Loitsiansky's invariant.

(IX) Turbulent Flow with Velocity or Temperature Gradient

Thanks to the advancement of hot-wire technique, we have recently gained much experimental infor-

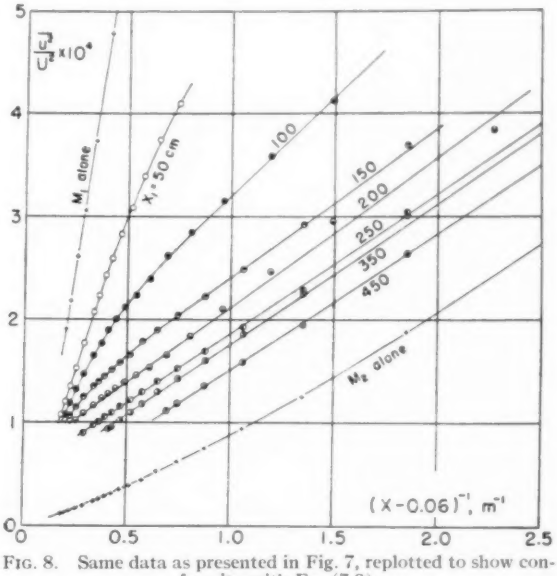

FIG. 8. Same data as presented in Fig. 7, replotted to show conformity with Eq. (7.3).

mation on turbulent shear flow, particular in the boundary layer.† However, no satisfactory theory of turbulent motion has yet emerged. The older theories of turbulent motion—such as the exchange coefficient and the mixture length theories—remain useful tools for practical applications. Recently, such theories have been applied to the turbulent boundary layer at supersonic speeds. Reference to such work will be found in the article on boundary layers.

Among the older theories of turbulent shear flow, von Kármán's similarity theory stands out as one permitting further refinement. The simple similarity concept proposed by von Kármán has not been completely verified experimentally, probably due to the existence of large eddies with a scale of the order of thickness of the boundary layer or the width of a channel. Still, it appears to this writer that the type of theory conceived by von Kármán deserves further attention. Either similarity may be regarded as a first approximation, or another type of relation may be found to connect the patterns of turbulent flow at two points in the same field. Recently, one uncertainty in the similarity theory has been removed by the use of Kolmogoroff's theory, and efforts have been made to extend the similarity theory to the case of compressible fluids.[54, 55]

One of the main purposes of Taylor's statistical theory was the clarification of the concept of the mixture length. Recently, Corrsin[56] applied such discussions to the study of the transfer of heat in a field with uniform temperature gradient. Studies of homogeneous turbulence in a field of uniform velocity gradient were made by F. Reis[57] and later by Burgers and Mitchner[58] with almost identical assumptions and results, although the work appears to be independent of each other. Further developments in this direction are still needed.

A more direct statistical approach to actual problems of shear flow was initiated by von Kármán,[3] developed by Chou,[59, 60] and extended further by Rotta.[61] The basic idea is the construction of statistical correlations in shear flow, mainly at a single point. Theoretically, it appears feasible to consider multipoint correlations and use approximate relations like the quasi-Gaussian assumption to reduce the number of unknowns down to the number of equations available. Such a scheme is no doubt very complicated, but it appears to be a reasonable approach to the difficult basic problem of turbulent shear flow.

(X) Other Aspects of Turbulent Flow

There are many other aspects of turbulent motion which have been more or less extensively studied—e.g., turbulent diffusion, atmospheric turbulence, the interaction between turbulent motion and compression waves (acoustic and shock), scattering of electromagnetic waves due to turbulent motion, and turbulent motion of an electrically conducting fluid in a magnetic field. Only some references to recent articles on these subjects are cited in the list of references.[62–71]

† See P. S. Klebanoff, *Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient*, NACA TN 3178, 1954, and the references quoted therein.

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Flame Theory and Combustion Technology

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SUMMARY

The study of combustion processes is in a sufficiently early stage so that there is no strong connection between combustion theory and the technology of combustion chamber development. To clarify such a connection is the principal task of workers engaged in establishing combustion as an engineering science.

The equations of aerothermochemistry are reviewed for the case in which temperature and composition gradients are small. Solutions have been obtained in very few cases and under very restrictive circumstances; most detailed considerations are restricted to the plane laminar flame front. The current situation in the theory of plane laminar flames is discussed. The few extensions that have been made to two-dimensional problems are then described. Several directions of work which would assist in establishing theoretical results approaching technological requirements appear possible.

INTRODUCTION

BEFORE THE STUDY of combustion processes can assume its position as a branch of engineering science, it is necessary to clarify the relationships between the aerothermodynamic and chemical-kinetic fundamentals underlying combustion and the complex processes that take place in combustion chambers. According to the spirit of engineering science, it need not be expected that the performance of combustors shall be predicted from fundamentals but, rather, that the thread connecting fundamental processes on the one hand with combustion technology on the other shall be thoroughly sound. Only then may valid approximations be made, appropriate experimental results introduced, and proper similarity criteria developed.

At the moment, this ideal seems to be a long way from realization. The instances where a realistic combustion process may be deduced from first principles are singular indeed. Furthermore, the fundamental information is not always available. Chemical kinetics is still unable to cope with many complex reactions of technological interest; fluid dynamics does not yet present a picture of the turbulence mechanism adequate enough to infer its influence on combustion processes; and basic experimental difficulties have rendered progress slow in obtaining reliable detailed information on combustion systems. On the one hand, there exists considerable information concerning the performance of combustion chambers for jet propulsion units; on the other hand, there has been considerable scientific study into the description of combustion details, particularly that of the laminar flame in combustible gaseous mixtures through application of chemical and thermodynamic fundamentals. But at this time there exists no clear connection between the two. This state is one through which most fields of engineering science have passed at one time or another during their development. Consequently, the present circumstances should not cast shadows on future possibilities. Aeronautical engineers are still aware of the period when the theory of ideal fluid motion appeared as a monstrosity having little bearing on the observed phenomena.

Under these circumstances, it is to be expected that Dr. von Kármán would, during recent years, devote a portion of his time to the clarification of this field of engineering. His interest is a natural outgrowth of his early and extensive work in the jet propulsion sciences. The impact of Dr. von Kármán's efforts in this field has been twofold. First, he has contributed his own deep insight and clear thought to formulation of key problems in the bridge between fundamentals and practice. Second, he has attracted the enthusiastic participation of workers disciplined in aerothermodynamics to problems formerly of interest principally to physical chemists. In assuring this cooperation between practitioners of the individual sciences and instructing them by example of the directions in which to work, it seems probable that Dr. von Kármán is at present accomplishing the essential step in developing aerothermochemistry into a mature engineering science.

The following paper describes briefly some of the contributions made by Dr. von Kármán and his collaborators and attempts to show how they fit into the present state of combustion theory. The author is indebted to his colleague, Prof. S. S. Penner, for advice concerning the contents of this review. In particular, the author acknowledges his deep gratitude to Dr. Hsue-shen Tsien from whose conversations many ideas have been drawn. Without the benefit of this material, the author's interpretation of Dr. von Kármán's current efforts would be far more faulty than it is.

AEROTHERMODYNAMIC AND CHEMICAL FUNDAMENTALS

Combustion processes are described by the motion of nonuniform gaseous mixtures in which chemical reaction produces concentration gradients and, consequently, diffusion of chemical species. Following the methods of Chapman and Cowling[1] or of Hirschfelder, Bird, and Curtiss,[2] appropriate mean values for the velocity, density, temperature, etc., of the gas mixture may be defined so that a gross continuity equation for the gas mixture may be written in the usual fashion

$$(\partial \rho / \partial t) + \text{div}\ (\rho \boldsymbol{V}) = 0 \qquad (1)$$

Likewise, in the absence of external forces, the Navier-

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Stokes equations of motion assume their familiar form

$$\rho(\partial \boldsymbol{V}/\partial t) + (\rho \boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\text{grad } p + \text{div } \tau \quad (2)$$

With the exception that τ is here the stress tensor for the nonuniform gas mixture; hence it includes contributions which arise from the fact that velocities of individual gaseous components differ from the mean velocity because of differential diffusion rates.

In dealing with the principle of energy conservation, it is convenient to introduce the concept of the total gas enthalpy, defined as the sum of the usual thermodynamic enthalpy $\int_{T_0}^{T} C_p dT$ and the enthalpy $h^{(0)}$ required to form the chemical species from its constituents at a fixed temperature T_0. Then, for a given chemical species, the total enthalpy h_i may be written

$$h_i = \int_{T_0}^{T} c_{p_i} dT + h_i^{(0)} \quad (3)$$

while the enthalpy for a gas mixture having a fraction Y_i of its weight contributed by the ith chemical species is

$$h = \sum Y_i h_i = \sum Y_i \left\{ \int_{T_0}^{T} C_{p_i} dT + h_i^{(0)} \right\} \quad (4)$$

the sum being taken over all chemical species. Now in a process involving an exothermic chemical reaction, the heat evolved arises from molecular binding energy released during the chemical rearrangement, and, clearly, this appears as a change in energy of formation between initial and final chemical species. Consequently, the heat associated with an exothermic reaction is explicit in this definition of the enthalpy. The energy conservation relation appears then as

$$\rho \frac{\partial}{\partial t}\left(h + \frac{|\boldsymbol{V}|^2}{2}\right) + (\rho \boldsymbol{V}\cdot\nabla)\left(h + \frac{|\boldsymbol{V}|^2}{2}\right) + \nabla \cdot \sum \rho Y_i h_i \boldsymbol{V}_{d_i} - \frac{\partial p}{\partial t} = \text{div } (\lambda \text{ grad } T) + \Phi \quad (5)$$

Aside from the definition of the enthalpy h, the only unusual term appearing is the divergence of $\sum(\rho Y_i \boldsymbol{V}_{d_i})h_i$. The mass transport of the ith species due to its diffusion velocity $\boldsymbol{V}_{d_i}$ is just $\rho Y_i \boldsymbol{V}_{d_i}$ and, consequently, the enthalpy transport is $(\rho Y_i \boldsymbol{V}_{d_i})h_i$ and, consequently, the enthalpy transport is $(\rho Y_i \boldsymbol{V}_{d_i})h_i$. The total enthalpy transport due to diffusion is then simply the sum of the individual transports $\sum(\rho Y_i \boldsymbol{V}_{d_i})h_i$. The quantity Φ is the viscous dissipation given in terms of velocity gradients and viscosity coefficients. Actually, this energy equation is correct for only terms linear in the diffusion velocities, and this restriction has been considered by von Kármán[3] whose results are quoted by Penner.[4] It appears that these higher order terms have not been deduced directly from the Boltzmann equation.

The gross continuity equation is not adequate to describe the conservation of each chemical species. For individual species, the motion differs from the mean motion by the local diffusion velocity $\boldsymbol{V}_{d_i}$, and the species is manufactured locally at the rate w_i. Then clearly

$$(\partial/\partial t)(\rho Y_i) + \text{div}[\rho Y_i(\boldsymbol{V} + \boldsymbol{V}_{d_i})] = w_i \quad (6)$$

and in virtue of the gross continuity relation of Eq. (1)

$$\rho\left(\frac{\partial Y_i}{\partial t} + \boldsymbol{V}\cdot\text{grad } Y_i\right) + \text{div } (\rho Y_i \boldsymbol{V}_{d_i}) = w_i \quad (7)$$

The diffusion velocity $\boldsymbol{V}_i$ arises from gradients in concentration, pressure, and temperature. In most combustion problems, the variations in pressure are sufficiently small so that this effect may be neglected. The diffusion caused by temperature gradients or thermal diffusion depends upon the existence of large temperature gradients and large differences of molecular weight between the interdiffusing species. A very interesting physical description of thermal diffusion is given by Jost.[5] Usually, but not invariably, this effect may be neglected in combustion problems and will be discounted in the following considerations. Then it is not difficult to show that, to the first-order perturbation of the Boltzmann distribution, the diffusion velocities of the ith and jth species are related as

$$\sum_j \frac{Y_i Y_j}{M_i M_j}\left(\frac{\boldsymbol{V}_{d_j} - \boldsymbol{V}_{d_i}}{D_{ij}} + \frac{1}{Y_i}\text{grad } Y_i - \frac{1}{Y_j}\text{grad } Y_j\right) = 0 \quad (8)$$

where D_{ij} is the binary diffusion coefficient between the ith and jth species, and M_i, M_j are their corresponding molecular weights. In addition, it is clear that the momenta associated with the diffusion velocities must sum to zero, and hence

$$\sum Y_i \boldsymbol{V}_{d_i} = 0 \quad (9)$$

The rates of production w_i of the various chemical species are governed by the laws of chemical kinetics. The gross chemical reaction is described by the numbers ν_i of initial atoms or molecules that react to produce ν_i' of the product molecules. If a chemical species is denoted by the symbol C_i, and all species entering into either the reactants or products are ordered $C_1 \ldots C_i \ldots C_N$, then the mass balance of atoms is written

$$\sum_1^N \nu_i C_i \rightarrow \sum_1^N \nu_i C_i \quad (10)$$

This is known as the stoichiometric relation; ν_i and ν_i' are the stoichiometric coefficients. Relations of this sort are usually inadequate to describe the detailed course of a reaction because the atoms or molecules of the left-hand side do not produce the final products through elementary collisions. Rather, they produce intermediate products which, in turn, react in a complex manner to produce the final products. Since the rates of reaction are related to the probability of "successful" molecular collisions, the entire set of elementary reactions must be known as

$$\sum \nu_i^{(k)} C_i \rightarrow \sum \nu_i^{(k)\prime} C_i \quad (11)$$

The overall result of Eq. (11) is the gross stoichiometric relation given by Eq. (10).

Associated with each elementary step is a rate at which it is carried out. For a given system, the probability of a successful collision depends upon (a) the fraction of molecules having sufficient energy—i.e., temperature—to engage in a successful process, and (b) the concentration of reacting species. The rate of a simple reaction may be expressed according to the law of mass action which states that the rate is proportional to the concentrations of reacting species. Thus, if the stoichiometric balance contains a species C_i, ν_i times, the reaction rate contains the concentration $[C_i]$ as $[C_i]\cdot[C_i]\cdot[C_i]\ldots \equiv [C_i]^{\nu_i}$. Consequently, the reaction rate may be written for the kth equation of type (11)

$$r^{(k)} \prod_i [C_i]^{\nu_i^{(k)}} \tag{12}$$

where $r^{(k)}$ is the specific reaction rate constant giving the probability for success of a given molecular encounter. Early in the study of chemical reaction rates, Svante Arrhenius[6] indicated that the specific reaction rate could be written

$$r = BT^{\alpha} e^{-E/RT} \tag{13}$$

where the activation energy E, the exponent α, and the frequency factor B depend upon the details of the particular reaction.

These basic processes, recently reviewed and clarified by von Kármán,[7] describe the details of ordinary aerothermochemical processes. The relations hold under the general assumptions that the Maxwell-Boltzmann distribution is not severely disturbed by the processes and that the changes of state encountered are sufficiently slow that equipartition of energy among all internal degrees of freedom of the molecule is satisfied at all times.

The Laminar Flame

One of the simplest problems arising from the complex aerothermochemical system described in the previous section is the plane flame front in a uniform parallel flow of a laminar gas stream. Consider the gas flowing in the direction of positive x axis; it is desired to calculate the gas speed required to produce a stationary reaction zone normal to the direction of flow and to determine some details of the reaction zone. This problem has been considered a classical one since the pioneering investigations of Mallard and Le Chatelier;[8] an excellent survey of the early theoretical attack on the problem has been given by Evans.[9] For this one-dimensional steady process, the gross continuity condition, Eq. (1), reduces simply to $\rho u = m = \text{constant}$. The momentum equations reduce to a single one that relates the pressure gradient in the direction of flow to the stream velocity and velocity gradient. It may be shown, a postiori, that the normal velocities of flames in laminar gas mixtures are so low that the resulting pressure variations are negligible. Hence, the momentum equation need not enter in the analysis.

The fact that the flame propagation velocity is low also introduces simplifications into the energy conservation relation. In the first place, the work done against the infinitesimal pressure gradients may be neglected; in the second place, the dissipation, equal to $(4/3)\mu \times (\partial u/\partial x)^2$, may be dismissed as unimportant in comparison with heat transferred by conduction. Then it is clear, from this simplified energy relation, continuity equations for individual species, and knowledge of the chemical-kinetic processes, that the details of the reaction zone may be expressed as distributions of temperature and chemical species. Boundary conditions are easily established on the problem since initial temperature and composition is given, and final temperature and composition may be determined from chemical equilibrium calculations. Furthermore, no heat or chemical species may be conducted or diffused from the field at points far upstream and downstream of the flame zone. Consequently, it is required that

$$\left.\begin{aligned} T &= T_0\,; \quad (dT/dx) = 0 \\ Y_i &= Y_{i,0}\,; \quad (dY_i/dx) = 0 \end{aligned}\right\} \quad \text{at } x = -\infty$$

$$\left.\begin{aligned} T &= T_f\,; \quad (dT/dx) = 0 \\ Y_i &= Y_{i,t}\,; \quad (dY_i/dx) = 0 \end{aligned}\right\} \quad \text{at } x = \infty$$

A certain difficulty is immediately apparent. If the initial reactable mixture exists at a temperature T_0, then, according to the specific reaction rate of Arrhenius [Eq. (13)] the mixture will be reacting at a finite rate. Unless the initial gas temperature has the unlikely value of absolute zero, the spatial variations of composition and the consequent heat release make it impossible to satisfy the conditions $(dT/dx) = (dY_i/dx) = 0$ far upstream of the flame zone. This essential incompatibility of boundary conditions and chemical-kinetic principles is known as the problem of the "cold boundary." The mathematical implications of this were discussed by Emmons[10] and by Adamson;[11] they investigated what must necessarily be prescribed far ahead of the flame to make the problem soluble and indicated that the flame itself was not too sensitive to conditions far upstream. In an effort to reconcile physical reality with mathematical accuracy, Hirschfelder and his collaborators[12] introduced an artificial "flame holder" endowed with the properties of absorbing heat and certain chemical species so that the flame could exist and still not alter the prescribed conditions far ahead of the reaction zone.

Later, von Kármán and Millan[13] discussed the cold boundary difficulty in detail for a reaction zone without diffusion. They first showed that the existence of a fixed flame speed in the mixture was equivalent to assuming an ignition temperature T_i—that is, a temperature below which no chemical reaction takes place. Thus, the most involved devices introduced to force a solution to the cold boundary problem were finally no more penetrating than the ignition temperature concept employed by the pioneers in the field. Of more importance to the solutions of combustion problems was their demonstration that the calculated flame speed was quite insensitive to the value assumed for the ignition temperature over a surprisingly wide range that depends upon the chemical details of the reaction. This confirmed the assumption made by Boys and Corner[14] that a good esti-

mate of the flame speed could be made by an approximation accurate near the hot boundary of the flame and relatively inaccurate at the cold boundary.

The technique for approximate integration of the aerothermochemical relations starting from the hot flame boundary was extensively developed by von Kármán and Millan[13] for a system described by a single global reaction without diffusion. The extension to more complex systems of chemical kinetics and to flames with diffusion of chemical species has been undertaken by von Kármán and Penner[15, 16] for two examples where reasonable knowledge of the kinetic details is known. Clearly, as the number of important elementary steps [Eq. (11)] increases, the problem becomes less tractable analytically. For complex systems, the only possibility for reasonable solution lies in simplifications that may arise because some of the elementary reactions are very fast while others are relatively slow.

To illustrate the simplifications that may be made in this way, consider the kinetics of ozone decomposition which were used by von Kármán and Penner[16] in their most detailed example. The decomposition of O_3 proceeds according to the elementary steps

$$O_3 + (M) \rightarrow O_2 + O + (M) \tag{14}$$

$$O_2 + O + (M) \rightarrow O_3 + (M) \tag{15}$$

$$O_3 + O \rightarrow 2O_2 \tag{16}$$

$$2O_2 \rightarrow O_3 + O \tag{17}$$

$$2O + (M) \rightarrow O_2 + (M) \tag{18}$$

$$O_2 + (M) \rightarrow 2O + (M) \tag{19}$$

where (M) indicates any massive body that does not take part in the reaction. In Eq. (14), the decomposition of ozone proceeds at a rate $10.56 \times 10^{12} (p/RT) \times \sqrt{T}\, X_3 \exp(-24140/RT)$, while the reverse reaction, Eq. (15), goes with a rate $0.23 \times 10^{12} (p/RT)^2 \sqrt{T}\, X_1 X_2$. The chain-breaking reaction given by Eq. (16) proceeds at a rate $7.15 \times 10^{12} (p/RT) \sqrt{T}\, X_1 X_3 \exp(-6000 \div RT)$, while its reverse, Eq. (17), has a rate $2.93 \times 10^{12} \times (p/RT) \sqrt{T}\, X_2^2 \exp(-99210/RT)$. Finally, the association of oxygen atoms, Eq. (18), has a rate $0.482 \times 10^{12} (p/RT)^2 \sqrt{T}\, X_1^2$, while the dissociation of oxygen molecules follows the law $8.92 \times 10^{12} (p/RT) \sqrt{T}\, X_2 \times \exp(-117350/RT)$. Here the mole fractions of O, O_2, and O_3 have been denoted X_1, X_2 and X_3, respectively. No effective global reaction is obvious largely because of the significant role played by oxygen atoms, the chain carrier.

After its manufacture, during the decomposition of ozone—according to Eq. (14)—the oxygen atoms enter again into reaction with ozone—according to Eq. (16)—to produce additional molecular oxygen. Through a little analysis, an effective global reaction may be found. For, when the mole fraction X_1 of atomic oxygen is small while the mole fractions X_2 and X_3 of molecular oxygen and ozone, respectively, are not small, the decomposition of ozone—according to Eq. (14)—strongly dominates its inverse given by Eq. (15). Similarly, due to the low activation energy for reaction of ozone with atomic oxygen—according to Eq. (16)—this reaction predominates over its inverse, Eq. (17), provided only that $X_1 \gg 10^{-19}$. Between the dominant reactions given by Eqs. (14) and (16), atomic oxygen is manufactured and consumed at about the same rates; for, if O accumulated, it would retard the first reaction and accelerate the second until equilibrium was restored. Conversely, the second reaction would be retarded if the supply of atomic oxygen became depleted. Since these two reactions take place much faster than the dissociation of oxygen, Eqs. (18) and (19), at flame temperatures, it is clear that the rates of manufacture and utilization of atomic oxygen are nearly equal over that portion of the flame where X_2 and X_3 are of order unity while $10^{-3} > X_1 \gg 10^{-19}$. Therefore, with good approximation, the two rates are equal.

$$10.56 \times 10^{12} \sqrt{T}\, X_3 \exp(-24140/RT) \approx 7.15 \times 10^{12} \sqrt{T}\, X_1 X_2 \exp(-6000/RT)$$

which determines the concentration of oxygen atoms as

$$X_1 \approx (10.56/7.15) \exp(-18140/RT) \tag{20}$$

For temperatures of interest in flame studies, this value of X_1 satisfies the restrictions $10^{-3} > X_1 \gg 10^{-19}$, and, consequently, the approximate balance above is justified. This result means that for every O_3 molecule that decomposes—according to Eq. (14)—an oxygen atom is produced which destroys another O_3 molecule—according to Eq. (16). Hence, the rate of disappearance of ozone is just twice the rate of the reaction given by Eq. (14), and, since this rate involves only the concentration of O_3, it will suffice as the global reaction treated by von Kármán and Millan.

This analysis, given here in a rather intuitive form, is carried out in detail by von Kármán and Penner.[16] They observe, furthermore, that this condition constitutes simply an example of the so-called steady-state hypotheses—introduced by Bodenstein,[17] and now familiar to all chemists—that the rates of change of some intermediate products may be neglected in comparison with rates of change of primary constituents. The success of its application to the ozone flame has encouraged von Kármán and his collaborators, as well as other investigators,* to explore how far this principle may be used to simplify further examples in the theory of laminar flames.

Toward the hot boundary of the flame, however, the ozone concentrations become so small that the decomposition of ozone—according to Eq. (14)—comes nearly to a balance with its inverse. Then the decomposition of ozone proceeds according to Eq. (16), consuming the available supply of atomic oxygen until both O_3 and O approach their equilibrium concentrations at the adiabatic flame temperature. Thus, the last small portion of the reaction proceeds at roughly half

* Recently M. Gilbert and D. Altman of the Jet Propulsion Laboratory at California Institute of Technology have been investigating its application to the Hydrogen-Bromine flame.

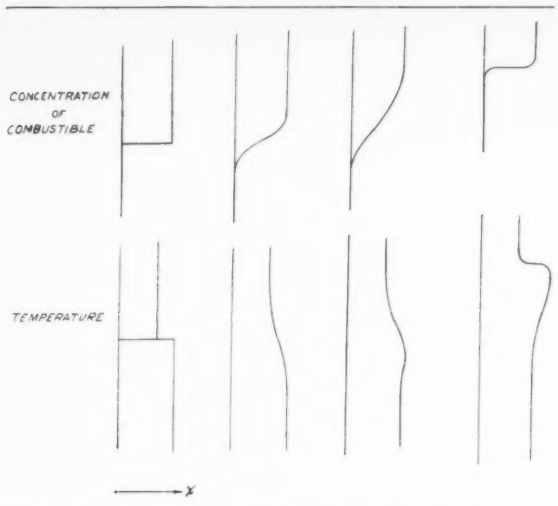

FIG. 1. Development of temperature and concentration profiles during ignition in a laminar mixing zone.

the rate of the larger part. It is not difficult to show that this final portion accounts for less than 1 per cent of the temperature rise and, consequently, is not influential in determining the flame speed. The more important consequence of this change in the kinetic process toward the end of the flame zone is that the reactions taking place at the hot boundary are not representative of those in the bulk of the flame zone. Therefore, an approximation beginning at the hot boundary with the conditions there would be grossly in error for the principal flame zone. Accurate application of the approximate integration techniques of Boys and Corner or of von Kármán and Millan require recognition of the differences between this thin, hot boundary zone and main flame region.

Laminar Combustion and the Boundary Layer

Combustion processes of a two- or three-dimensional character may be constructed when the combustion zone is replaced by a discontinuity imparting a fixed temperature rise to the gas and moving into the unburned gas with a fixed normal velocity. Such flow fields have been constructed by Tsien,[18] by Barrere and Mestre,[19] and by Gross and Esche[20]. This approximation provides an adequate description when the flame radii of curvature are large in comparison with the flame thickness and when no strong temperature gradients, composition gradients, or other conditions are imposed which upset seriously the local structure of the flame zone. When the combustion front is not uniform over its length, the problem becomes very difficult since the one-dimensional analysis described previously cannot be applied locally. However, with the knowledge gained through analytic solution of the plane flame front and the use of approximation techniques introduced by von Kármán, some progress may be made in discussing some particular cases of the true two-dimensional combustion zone.

Investigation of these problems is facilitated by introduction of the boundary-layer approximation. As a matter of fact, the preponderant number of technically important two-dimensional combustion zones originates in regions where the gradients of important quantities normal to the direction of flow are much larger than those in the direction of flow. Examples of this circumstance occur in the quenching of flames near cold surfaces, the erosive burning of solid propellants, the ignition of a combustible stream or jet by laminar mixing with an adjacent stream of hot gas moving at a different velocity, etc. Employing the boundary-layer approximation and certain of the techniques used by von Kármán, the ignition and combustion in the laminar mixing zone between two streams was investigated by Adamson and the present author.[21] Let one stream be of combustible mixture at temperature T_1 flowing with velocity U_1, the other of reaction products at temperature T_2, $T_1 < T_2 < T_f$, moving with a velocity U_2. From the start of mixing at $x = 0$, the streams develop velocity, temperature and composition profiles due to simple shear, heat conduction and diffusion—Fig. 1. Farther downstream, sufficient combustible material has been heated so that the chemical reaction produces its own peculiar changes in the temperature and concentration profiles. At first the combustion zone is quite diffuse, but gradually it contracts into a zone resembling the laminar flame. Since the mixing zone grows roughly as $\sqrt{x}$, the reaction zone will eventually move out of the laminar mixing region and develop into a true plane flame front. If the velocities of the two gas streams are such that the laminar flame makes a very small angle with respect to the x axis, the boundary-layer approximation is applicable throughout the region of flame development.

Under the boundary-layer approximation, only dynamic equilibrium in the direction of flow need be considered, and, because of the low value of laminar combustion velocity, the pressure variations over the field are negligible. Therefore

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = (\partial/\partial y)[\mu(\partial u/\partial y)] \quad (21)$$

and the gross continuity equation is

$$[\partial(\rho u)/\partial x] + [\partial(\rho v)/\partial y] = 0 \quad (22)$$

The temperature field is described by the energy equation. If the enthalpy is separated into its thermodynamic and chemical portions, the energy equation may be written

$$\rho u(\partial/\partial x)(C_p T) + \rho v(\partial/\partial y)(C_p T) = (\partial/\partial y)[\lambda(\partial T/\partial y)] - qw \quad (23)$$

where q is heat released through consumption of a unit mass of combustible gas. Assuming a simple global reaction involving only the mass fraction of combustible mixture Y, the "rate of appearance" of combustible matter is

$$w = -(\rho Y/\tau) \exp(-E/RT) \quad (24)$$

where τ, the characteristic time of the reaction, is re-

lated to the frequency factor. By virtue of the gross continuity condition, Eq. (22), only the conservation of combustible species need be considered. If D is the diffusion coefficient of combustible gas into the combustion products, the continuity of the reactable species is given by

$$\rho u(\partial Y/\partial x) + \rho v(\partial Y/\partial y) = (\partial/\partial y)[\rho D(\partial Y/\partial y)] + w \quad (25)$$

The set of relations given by Eqs. (21) through (25), together with limiting conditions of velocity, temperature, and concentration in the two streams, describe completely the process of ignition in the laminar mixing zone between two parallel streams.

The initial development of the combustion process is calculated most easily by expansion in powers of x, the so-called Blasius series. The fact that this approximation breaks down in excess of a certain distance downstream from the start of mixing implies that a characteristic time or distance must be implicit in the problem. Certainly this is related to the characteristic time τ of the reaction; a detailed calculation of time required in this problem before a significant rate of heat release is achieved gives the characteristic time as

$$t = \tau(E/RT)[1 - (T_1/T_2)] \exp (E/RT_2) \quad (26)$$

so that it depends not only upon the time constant τ but very strongly upon the activation energy. The power series breaks down when $(x/u_1 t)$ becomes of the order unity. The solution is extended beyond this region through application of the integral technique introduced by von Kármán into the calculation of boundary-layer flows.

Another problem of great technical significance, flame-quenching near a cool surface, may be investigated with the aid of boundary-layer approximations. Although this problem is of historic interest in combustion theory and experiment, the first effort at its analysis from first principles was undertaken by von Kármán and Millan.[22] In addition to Eqs. (21) through (25), there are the boundary conditions: (a) no gas velocity normal to or parallel to the wall at the wall surface, (b) the temperature at the wall surface is constant and equal to T_w, and (c) no net diffusion of species into or away from the wall. This latter condition is equivalent to the assumption that the process is entirely one of thermal quenching—that is, the wall removes heat from the combustion process at a rate depending upon the local temperature gradient, but individual chemical species are reflected from the wall unmodified by the collision; no surface reaction takes place. This restriction on the formulation of this problem should be removed when chemical kineticists provide conclusive information on the processes taking place at the surface.

Von Kármán and Millan consider a normal combustion front moving into a stationary gas bounded by a constant temperature wall; the process is made stationary by allowing the wall to move at the speed of normal flame propagation. The fluid dynamic aspects are approximated by assuming the velocity normal to the wall to vanish uniformly over the field. They obtained the position of the flame front through an approximate calculation of the temperature field and assumed the flame to lie along the isotherm corresponding to the "ignition temperature" in the sense of their laminar flame calculation. Their results are shown in Fig. 2; the lengths are given in terms of the heat conduction length $(\lambda/\rho_0 u_0 C_p)$ so that the horizontal coordinate is $\xi = (\rho_0 u_0 x C_p/\lambda)$, while the vertical coordinate is $\eta = (\rho_0 u_0 y C_p/\lambda)$, and the temperature field as a ratio of local temperature to the flame temperature is $\vartheta = (T/T_f)$. The vertical isotherm at $\xi = 0$ corresponds to the normal flame in the free stream. The temperature field for $\xi > 1.5$ is adequately approximated by that which would exist if the normal flame extended to the wall.

The geometry of the isotherms joining these two limiting states was approximated by the integral techniques that von Kármán introduced into boundary-layer problems and which was applied successfully in the problem of ignition in a laminar mixing region. It is clear that a certain difficulty occurs in describing this portion of the field. For the normal flame, the gradients are predominantly in the direction of flow; for the region far downstream, the gradients are predominantly normal to the flow direction since this portion of the field is established by heat conduction to the wall. The transition region between the two limiting states does not satisfy the criteria for employing the boundary-layer approximation. Thus, Eq. (23) does not necessarily hold in the transition region. Von Kármán consequently included the thermal conduction $\lambda(\partial T/\partial x)$ in the direction of flow which is omitted in classical boundary-layer considerations. Aeronautical engineers will recognize here a situation reminiscent of that arising in the problem of shock wave-boundary layer interaction. The isotherm $\vartheta = 0.7$, corresponding to the ignition

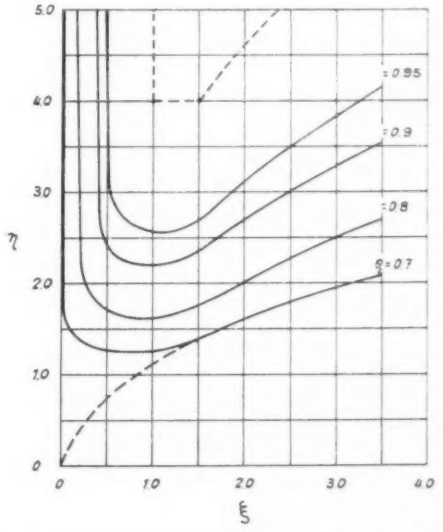

FIG. 2. Isotherms near a laminar flame front quenched by a cool wall.

temperature and hence the approximate position of the flame front, passes much closer to the wall than the usually accepted quenching distance. Numerical conclusions, however, can hardly be drawn since the chemistry employed was arbitrary and somewhat oversimplified, and the measurements cannot be considered precise.

Conclusion

Although these investigations that have been described fall far short of the needs of combustion technological practice, they do treat some of the key questions that arise in a wide variety of problems. Only through complete answers to some of these elementary questions will sufficient information be obtained to permit really sound similarity considerations. In a field so involved as combustion, it is likely that similarity rules will always constitute the most widely used results.

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strain rate but also upon time, the generalization of von Kármán's theory of propagation of plastic deformation to materials having a yield point appears to be a very difficult problem.

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Applied Aerodynamics and Flight Mechanics*

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SUMMARY

Applied aerodynamics and flight mechanics represent one of the many fields in which Dr. Theodore von Kármán has made major contributions. His work has been especially valuable because of his theoretical and applied research on problems useful to those concerned with the design of airplanes. Such work has dealt with skin friction at subsonic and supersonic speeds, transonic similarity, linear supersonic theory, hypersonics, superaerodynamics, and the cost of speed. His survey papers, covering various speed ranges on subjects now of prime importance and given prior to the time when applied design progressed to these speed regions, represent milestones of foresight and progress. His broad interest in flight mechanics and related problems of applied mechanics in other fields has given his work breadth of coverage in subject matter and applicability.

A design aerodynamicist has to consider carefully many aspects of each problem in order to achieve a well-balanced design. At subsonic speeds, turbulent skin friction comprises about 75 per cent of parasite drag and 50 per cent of total normal operating drag. It has been found by simultaneously considering important aerodynamic characteristics and certain simplified structural criteria that optimum unswept wings at supersonic speeds will be very thin and have very low aspect ratios. As at subsonic speeds, the zero lift (parasite) drag of these wings, as well as that of correspondingly slender bodies, will be composed principally of turbulent skin friction to which Dr. von Kármán's work continues to be applicable. Drag due to lift at supersonic speeds is many times that at subsonic speeds, leading to low ratios of lift to drag. Research to improve the drag due to lift at supersonic speeds is greatly needed. Improvements in the lift to drag ratio of wings can be largely reflected in similar improvements in complete airplanes as well.

The lower lift to drag ratio, although partially compensated for by better overall power-plant efficiency of turbojet engines at supersonic speeds, leads to much lower range for a supersonic transport. The past record would indicate that marked improvement in range can be expected with continuation of serious efforts in aeronautical research and design. Until considerable improvement in range can be made, the supersonic transport would be subject to the usual complications encountered when operating an airplane over routes requiring important flights longer than the maximum range of the airplane. A simple cost analysis shows that, up to the range attainable by a supersonic transport, there is good hope of retaining operating economy at supersonic speeds near costs attainable with jet transports at high subsonic speeds. This result is possible largely because of the higher speed, tremendous thrust output, and improved overall fuel efficiency of turbojet engines with afterburning.

As the speeds of vehicles increase to supersonic and hypersonic levels, their energy becomes very large and predominantly kinetic. In flight near the earth, speed will replace altitude as the major factor insuring effectiveness of action and operating efficiency. Range about the earth increases rapidly with speed, but the heating problem becomes very serious. In space flight beyond the earth's atmosphere, the aerodynamic problems proceed to those of the slip-flow region—it being conceivable that vehicles operating essentially in a vacuum will reach speeds expressed in terms of fractions (even though very small) of the speed of light. Dr. von Kármán's latest papers have dealt with heating, hypersonics, superaerodynamics, and high energy propulsion. His leadership here as before will most assuredly be applicable to the future problems of applied aerodynamics and flight mechanics in these new superspeed and superaltitude regions.

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INTRODUCTION

THIS PAPER IS presented in tribute to Dr. Theodore von Kármán. His contributions to applied aerodynamics and flight mechanics and their great influence on airplane design are viewed from the standpoint of one of his first students in America, who has made liberal use of these in the aerodynamic design of airplanes. Progress in this field is briefly traced, some speculations regarding the future are made, and the development of a general philosophy and procedures of the aerodynamicist in his approach to design are presented. Significant throughout is the influence of Dr. von Kármán's contributions and leadership.

OVERALL AERODYNAMIC DESIGN AND COMPROMISE

To one who is concerned with the design of airplanes, the goals of applied aerodynamics and flight mechanics are the achievement of optimum configurations from the standpoint of overall airplane design and operation. This situation might be deplored by the aerodynamic purist whose goal is that of achieving aerodynamic perfection alone, but the broad viewpoint is necessary to provide a realistic standard of values.

To illustrate, consider the choice of wing span. In the subsonic speed regime using aerodynamic criteria alone, one would most likely call for an infinite—or extremely large—span (hence, aspect ratio) in order to obtain the least possible drag for a given lift. Benefits such as long range, high ceiling, increased take-off weight, and other advantages would appear to result from the low induced drag associated with large span. Because of the increase in skin friction with decrease in Reynolds Number resulting from decreasing chord as span increases for a given wing area, one might be able to derive an optimum span (or aspect ratio) from aerodynamic considerations alone. However, this choice would be far from the practical choice resulting from properly balancing the aerodynamic gains against the structural weight effects. As a result of the design complications and weight introduced by very large span, the practicing aerodynamicist finds it most advantageous to arrive at a compromise in aspect ratio which, for today's subsonic designs, varies in magnitude

from about five to twelve depending upon speed and range requirements.

Conflicting requirements rather concisely define the configuration of an airplane designed to a certain specification. The airplane can be evaluated in a number of ways such as through considerations of speed, range, or pay load carried. The general similarity of airplanes being operated successfully by the highly competitive commercial airlines is quite significant, especially so since this similarity is largely the result of independent design teams working to achieve a given objective. Airplanes which have not been properly synthesized by considerations of all factors that make up the final product have not emerged as successful designs. The aerodynamicist concerned with airplane design can trap himself if he proceeds without properly considering the realities of practical design limitations.

While the other considerations of weight, power plant, and structure force a design compromise away from an aerodynamic optimum, the engineers in these fields concurrently are making refinements and innovations that help the aerodynamicist in obtaining the type of machine needed or desired. One of the great milestones in the progress of subsonic designs was the change from the biplane wing to the currently accepted cantilever monoplane wing. Initially, these wings were thick and straight, but to help in achieving subsonic speeds near the speed of sound, recent designs have thinner sections usually in combination with sweepback. In order to fly through the speed of sound into the supersonic speed regime, wings must be very thin. Important progress in structural design has made possible very thin wings of the order of 2- to 4-per cent thickness-chord ratio. With these wings, the aerodynamicist is assured of an efficient air frame that can be flown at supersonic flight speeds. Moreover, one can hardly imagine current aerodynamic progress in airplanes and missiles without the development of turbojet and rocket engines. Unless there can be obtained much better lift to drag ratios than are now available, much of any success of long-range flight at supersonic speeds will have to come from power plants highly improved in respect to thrust and fuel consumption.

Dr. von Kármán's Role in Applied Aerodynamics and Flight Mechanics

There is an intimate connection between Dr. von Kármán's work and the problems faced by the design aerodynamicist. Pioneering scientific leadership of highest caliber is essential to progress in aerodynamics and airplane design. In this era of growth to maturity of the airplane, we have been fortunate in having this leadership from Dr. Theodore von Kármán. Almost from the time of the first flight of the Wright Brothers, Dr. von Kármán has diligently applied his genius and generously disseminated his knowledge and works through papers, books, lectures to students, and personal work (advisory and otherwise) in the advancement of scientific and applied aerodynamics.

Dr. von Kármán's contributions are especially valuable to a design aerodynamicist in many specific ways. He has evolved theories on aerodynamic fundamentals, given surveys on timely subjects prior to their practical application, and contributed importantly in other related fields.

In the realm of aerodynamic theory, some of Dr. von Kármán's contributions to applied aerodynamics are clarifications and solutions of the problems of:

(a) Turbulent skin-friction variation with Reynolds Number.

(b) Turbulent skin-friction variation with Mach Number into the supersonic speed regime.

(c) Pressure coefficient variation with Mach Number in the subsonic speed regime.

(d) Transonic similarity rules for correlating information on thin bodies in the transonic speed range (Mach Numbers near 1).

(e) Nose shape for optimum pressure drag of slender bodies of revolution at supersonic speeds.

The great significance of his contributions is the bold and productive frontal attack on the most difficult problems of real interest to the designers of airplanes and solutions in the form of relations adequate for definite use in design. For example, he has helped to clarify the elusive concept of turbulence and produced useful theories and relations for turbulent skin friction. This is most fortunate because in spite of its complexity we have had to deal, in fact, with turbulent skin friction.

He has presented many survey papers of broad interest. Of special significance is the fact that they authoritatively covered subjects which were later to become of major importance. For example, papers were presented on:

(a) "Turbulence and Skin Friction"[1] in 1934—In this paper were given his logarithmic turbulent skin-friction law and the effects of surface roughness when the degree of roughness was of the order of the boundary-layer thickness. Timeliness of this development is evident upon realizing that it was contemporary with the end of the biplane era and the emergence of the smoothly cowled monoplane, for which skin friction is the predominant portion of parasite drag. Significant was the section on "Practical Applications."

(b) "The Problem of Resistance in Compressible Fluids"[2] in 1935—In this paper, there was proposed von Kármán's variation of turbulent skin friction with Mach Number at high subsonic and supersonic speeds. The optimum drag ogive was presented. Flow problems into the hypersonic range were discussed. This paper was well ahead of the applied aerodynamic art, preceding supersonic level flight by some 10 years. The subjects discussed are of prime importance today in supersonic airplane and missile design.

(c) "Turbulence"[3] in 1937—In this paper, the fundamental framework for the statistical theory of

turbulence was laid. Also covered was previous work on turbulence of interest to designers.

(d) "Compressibility Effects in Aerodynamics"[4] in 1941—This paper covered principally aerodynamic effects at high subsonic and transonic speeds and included the Kármán-Tsien hodograph method of deriving the variation of pressure with Mach Number, a method better than the Prandtl-Glauert rule and sufficiently accurate for most aerodynamics applications. The concept of effective thickness growth of thin wings with increasing subsonic Mach Number was presented. From 1941 to the present time, compressibility problems at high subsonic speeds have been of great importance; hence, the material of this paper was most timely.

(e) "Problems of Flow in Compressible Fluids"[5] in 1941—This paper contained a presentation of fundamental background and included his own important developments on the subject.

(f) "Faster than Sound"[6] in 1945—In this paper Dr. von Kármán, in answer to the question, "Shall we ever fly faster than sound?", stated his conviction that ". . .there is no evidence that the velocity of sound should constitute a 'stone wall' of despair. To be sure, it will be necessary to use all good advice which aerodynamic science, chemistry of combustion, and thermodynamics may contribute." How cautious, yet prophetic in assurance of accomplishment!

(g) "Supersonic Aerodynamics—Principles and Application"[7] in 1947—The design implications of the Mach cone, the concept of zones of action and silence and of wave drag, the cardinal rules of supersonic theory, the transonic similarity rule, and the applications of these principles to range prediction of supersonic airplanes were presented and discussed. This paper also directed attention to the invariance of drag with forward or reversed direction of flight for nonlifting wings in supersonic flight. This material has had an important influence on subsequent theoretical developments, as well as immediate practical utility in the computation of wing characteristics. The bibliography included over four pages of references. This paper was truly a forerunner of the era of supersonic flight which today is realized to be the new field in which future flying is to be done.

(h) "On the Foundation of High Speed Aerodynamics"[8] in 1954—In his chapter, Dr. von Kármán discusses hypersonic theory, viscosity and heating at high speeds, and superaerodynamics (the problems of motion of bodies through rarefied air, such as slip flow where the fluid no longer has zero velocity at the surface of a body). Here is presented authoritative discussion sure to be of help in the future regime of flight of airplanes, missiles, satellites, and interplanetary vehicles.

He has made major contributions in other related fields, particularly hydrodynamics, elasticity, applied mechanics, physics, combustion, and overall problems in the evolution of new designs. This wide interest leads to his unique evaluation of the trends of airplane designs and needs of designers and has directed his efforts to theories and problems of immediate or early real value. No wonder design aerodynamicists are grateful to the grand master of applied aerodynamic theory!

The Importance of Skin-Friction Drag

In the Wilbur Wright Memorial lecture of 1937,[3] Dr. von Kármán states, "Skin friction is perhaps the most important factor for the speed performance of a modern aeroplane. Some of our modern planes are aerodynamically clean to such an extent that over 60 per cent of their total drag* is supposed to be due to skin friction." Experience with designs currently flying confirms Dr. von Kármán's statement; in fact, it points to an even greater role played by skin friction. A further examination of the composition of drag of today's airplanes might be of interest at this point.

It is well-known that at the best ratio of lift to drag, $(L/D)_{max}$, the drag is equally divided between parasite drag, composed at subsonic speeds of pressure and skin-friction drags, and drag due to lift. Airplanes usually fly at attitudes varying from that corresponding to the speed for best lift/drag ratio, $V_{L/D}$, for longest range, to speeds much higher. Usually long-range flying is done at speed ratios, $k = V/(V_{L/D})$, from 10 to about 30 per cent above the speed for best L/D. The inverse of the ratio of (L/D) to $(L/D)_{max}$ at speed ratio k is

$$\frac{(L/D)_{max}}{(L/D)} = \frac{1}{2}\left(k^2 + \frac{1}{k^2}\right) \qquad (1)$$

The first term represents drag which is independent of lift, the second that due to lift.

For maximum range with a propeller driven airplane, the theoretical value of k is unity, the practical value being about $k = 1.10$. At $k = 1.10$, the parasite drag is nearly 60 per cent of the total. For jet airplanes operating at high subsonic speed for which thrust is insufficient for reaching the ideal altitude where speed for best L/D occurs simultaneously with the speed for drag rise (Mach divergence $M_{div.}$), maximum range is obtained at nearly 20 per cent above best L/D speed. Here parasite drag is about 68 per cent of the total. Thus, it is reasonable to assume that about two thirds of the total drag in cruising flight is parasite drag.

It is now of interest to inquire what fraction of parasite drag is skin-friction drag. This ratio is normally termed "cleanness factor." In Fig. 1 are plotted "cleanness factors" from flight tests of the principal Douglas DC-series airplanes and an early biplane. Two curves are shown. One curve is for cleanness factor, F_A, defined as the ratio of the turbulent skin-friction drag (calculated from von Kármán's logarithmic turbulent skin-friction law summed over all the

* His subsequent sentences indicate that by total drag was meant parasite drag only—i.e., skin friction plus form drag—without considering drag due to lift.

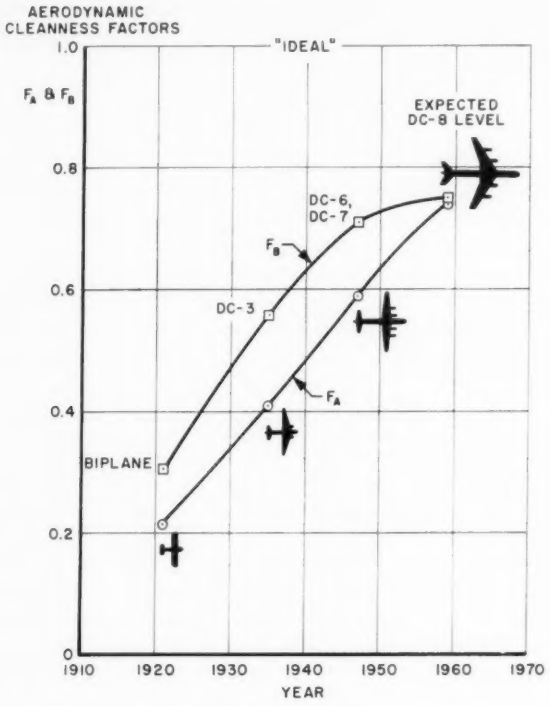

FIG. 1. Aerodynamic cleanness factor versus year.

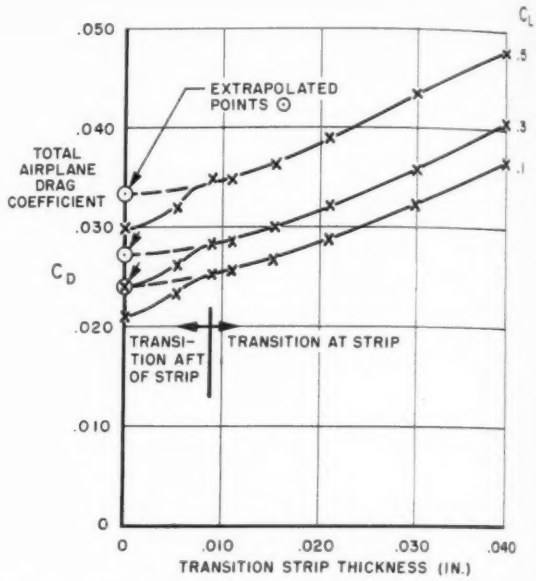

FIG. 3. Drag coefficient from test with turbulent boundary layer fixed by transition strips of varying thickness.

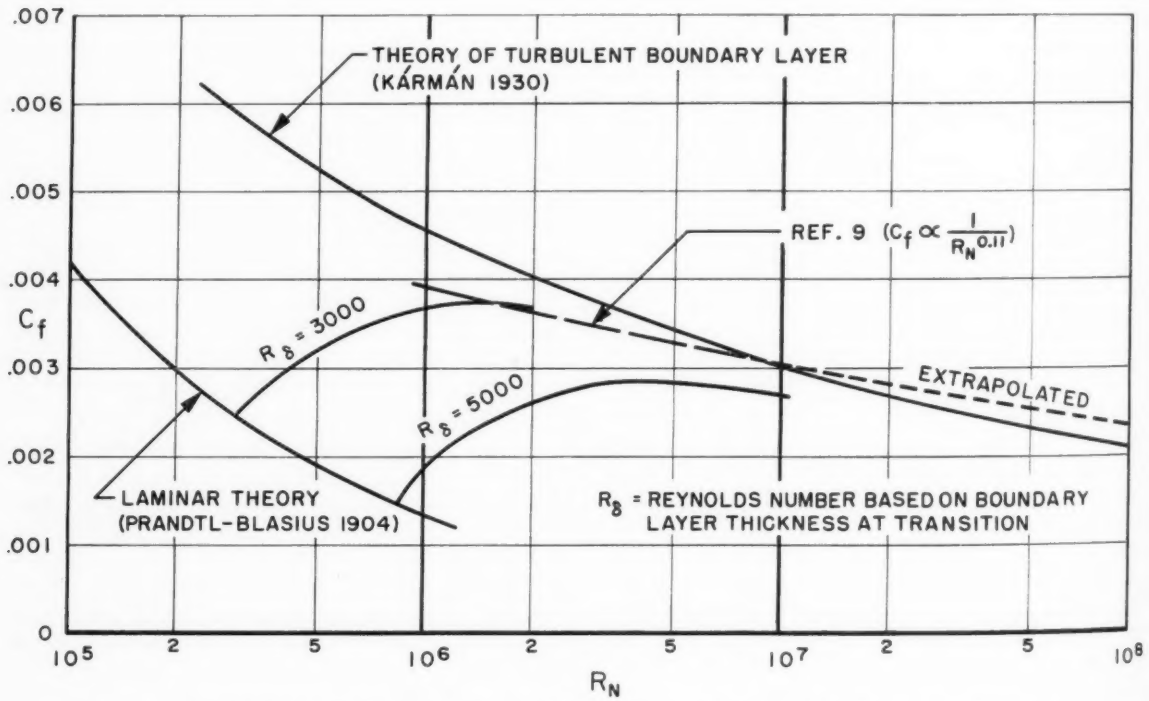

FIG. 2. Mean skin-friction coefficient versus Reynolds Number (based on reference 1, Fig. 12, and reference 9).

parts) to the total measured parasite drag of the airplanes.

$$F_A = (\Sigma C_f W_A)/f \tag{2}$$

where

C_f = mean coefficient of skin friction of each part at its Reynolds Number

$R_N = \rho V L/\mu$ = Reynolds Number of each component part based on a mean length, L

ρ = air density

V = velocity

μ = viscosity of air

W_A = wetted area of each component part

f = total parasite drag area = $\dfrac{\text{parasite drag}}{\text{dynamic pressure}}$

$= \dfrac{D}{q}$

$= C_{D_P} S$ where C_{D_P} = parasite drag coefficient, S = wing area

$q = \rho V^2/2$

Cleanness factor F_B is defined as

$$F_B = \frac{\Sigma C_f W_A}{f \text{ (excluding the drag areas associated with engine cooling and known specific items such as radio, landing gear, leakage, etc.)}} \tag{3}$$

and a curve drawn for this factor is shown. Cleanness factor, F_A, is the one customarily used, but cleanness factor, F_B, gives a better assessment of surface cleanness by subtracting known drags not associated with surface condition.

It has been design practice to use directly, for bodies, the von Kármán logarithmic law for skin friction shown in Fig. 2 and given by the relation[1]

$$1/C_f^{1/2} = 1.7 + 4.15 \log_{10} (R_N C_f) \tag{4}$$

An increment in drag coefficient is allowed for surface condition which depends upon the type of construction and surface finish, and a form drag coefficient increment dependent on body fineness ratio is also added.

For wings, a slight modification to the trend of Eq. (4) has been used as shown in Fig. 2. This curve was based on NACA tests made in the Reynolds Number range of transition from laminar to turbulent flow.[9] It is now believed that at the higher full-scale Reynolds Numbers, the von Kármán variation should be closely followed. Increments in drag coefficient for surface condition and thickness ratio are added.

Reference to Fig. 1 shows that the overall wetted area cleanness factor F_A has increased from about 22 per cent for the biplane to 41 per cent and 58 per cent, respectively, for the DC-3 with brazier-head rivets and the flush-riveted DC-7. For the heavily skinned, jet-powered DC-8, it is estimated that F_A will increase to 74 per cent. Corresponding values for F_B are 30 per cent for the biplane, 56 per cent for the DC-3, 71 per cent for the DC-7, and 75 per cent for the DC-8. Thus, with modern airplanes, an even greater portion of drag is now skin friction than the 60 per cent previously estimated by Dr. von Kármán.

It is apparent that if about two thirds of the total drag in cruising flight is parasite drag of which 75 per cent is now skin friction, then approximately one half of the total resistance of the latest airplanes is purely turbulent skin friction, for which von Kármán's theory may be used for design purposes.

The problem of estimating drag from wind-tunnel tests, or otherwise, is important to the designer. Low-scale wind-tunnel tests have formerly been almost worthless for drag estimates because the test Reynolds Numbers usually fell in the region of transition from laminar to turbulent flow. It is customary to estimate parasite drag by methods developed from the correlation of flight tests of previous full-scale airplanes. In order to improve our technique of estimating the drag of airplanes from wind-tunnel data for design purposes, a method has been investigated by Dunlap* in which transition is fixed by strips of various thicknesses to insure having turbulent flow in the same regions as expected on the full-scale airplane. From tests in the tunnel, there can be extrapolated a value of drag with zero thickness of transition strip for the smooth model with turbulent boundary layer as shown in Fig. 3. After proper consideration of drag due to lift from tests at various lift coefficients, further extrapolation to full scale Reynolds Number from von Kármán's curve should enable making a good estimate of the skin-friction drag of the airplane. Total drag area, f, is then obtained by adding estimated drag for cooling, leakage, protuberances, form, roughness (which today is becoming very small), etc. This method of obtaining a test point of the scale model essentially on the fully turbulent curve gives hope of making reasonable extrapolation to full scale.

The Composition of Supersonic Drag of an Optimal Design

As pointed out in various papers of Dr. von Kármán, linear supersonic theory is well advanced. This theory has shown that there are two additional types of drag at supersonic speeds: wave drag due to thickness and wave drag due to lift. For thickness ratios like those used at subsonic speeds, the zero-lift wave drag is prohibitively high. It is discouragingly large even for thickness ratios of, say, half the values usually encountered in subsonic airplanes. Still thinner wings require a reduction in aspect ratio for structural reasons, which increases the usual vortex drag due to lift. Fortunately, however, the wave drag due to a given lift decreases with increasing chord—so that the low aspect ratio necessary with the use of very thin

* Unpublished investigations by R. A. Dunlap, Aerodynamics Section, Engineering Department, Santa Monica Division, Douglas Aircraft Company, Inc.

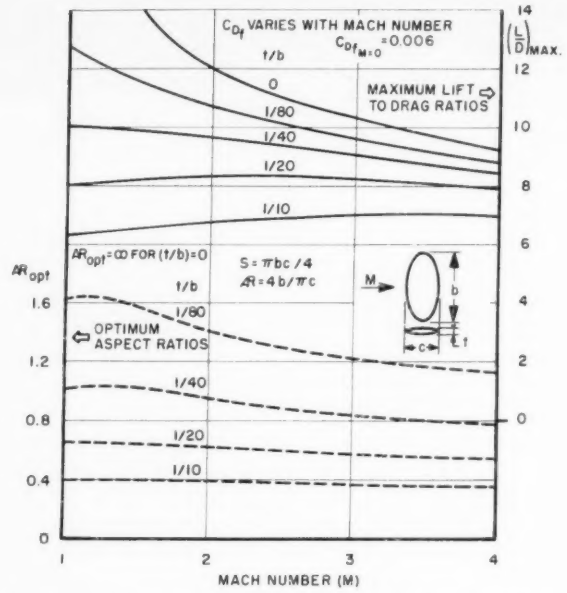

Fig. 4. Aspect ratios and maximum lift to drag ratios of optimum elliptic plan forms versus Mach Number for constant ratios of root thickness to span (from reference 10).

wing sections also helps in reducing at least one part of the drag due to lift.

A sensible optimum cannot be arrived at from aerodynamic considerations alone. The aerodynamic benefits of large span and chord (required for low drag due to lift) and extremely thin sections (to minimize the drag due to thickness) must be weighed against the penalties imposed by the structure. An analysis of the problem in which structural considerations were introduced was made by Graham and Licher.[10] They found optimum aspect ratios as a function of flight Mach Number, on the basis of chosen structural criteria, considering wave and vortex drags due to lift, wave drag due to thickness, and turbulent skin-friction drag. To simplify the analysis, unswept elliptical wings were assumed. Several simplified structural criteria were studied. The principal results of their study are given in Fig. 4 which shows the variation of optimum aspect ratio and $(L/D)_{max}$ with Mach Number for various ratios of t/b, the median criterion of those studied. Investigation of many designs has shown that the ratio of root thickness to span, t/b, falls in a fairly narrow band for most successful designs.

The variation of the turbulent skin friction with Mach Number was taken for the study to be that shown by the solid line in Fig. 5, which is based on the skin-friction measurements of the cited references. Also shown are two other curves taken from an NACA Technical Note[11] which show variations of turbulent skin friction with Mach Number. One curve represents a calculation of von Kármán's relation for certain assumed conditions. The other curve represents a recent interpretation of available test data. The curves based on test data follow the trend indicated by von Kármán but to a somewhat reduced degree.

To illustrate the significance of the results of this analysis for optimizing wings, consider the following example at a Mach Number of 2.5. At this Mach Number, there is a reduction in skin-friction drag coefficient of about 40 per cent which has a considerable effect upon the final L/D obtained for the optimum wings. Fig. 4 shows that at an assumed thickness span ratio, t/b, of about $^1/_{50}$ the optimum L/D of the wing alone is about 9.7 at a Mach Number of 2.5, and the corresponding aspect ratio is about 1.0. The thickness ratio of the wing root is about 1.6 per cent. The optimum aspect ratio decreases only slightly with Mach Number and should be reduced to an unusually low value by present standards to effect marked reduction in the new drag terms with a structurally practical wing.

The proportions of the various types of drag in these optimized wings are shown in Fig. 6 for the attitude of maximum lift to drag ratio where 50 per cent of the drag is zero-lift drag (comparable to parasite drag at subsonic speeds) and 50 per cent that due to lift. Note that in this figure the abscissa can be considered to be roughly proportional to Mach Number at moderate supersonic speeds. It is almost shocking to note in this optimum wing of the chosen example, which gives consideration to a structural criterion, that the thickness drag is reduced to a minor portion, about one eighth of the parasite drag (one sixteenth of the total) at Mach Number 2.5. Hence, skin friction has again emerged as the predominant item of drag at zero lift. As Mach Number increases above 2.5, the zero-lift drag can become almost entirely that of turbulent skin friction for which we have von Kármán's theory as a guide for design at supersonic speeds.

Previously it was indicated that two additional types of drag make their appearance at supersonic speeds. The wave drag due to thickness was found to be very small, and hence, parasite drag is seen to be essentially skin friction for a realistic airplane design. The second additional type of drag, wave drag due to lift, poses a problem that has emerged as the serious one for flight at supersonic speeds. In the optimized wing, the drag

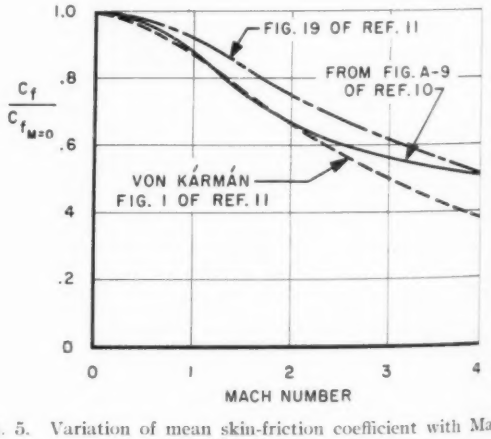

Fig. 5. Variation of mean skin-friction coefficient with Mach Number.

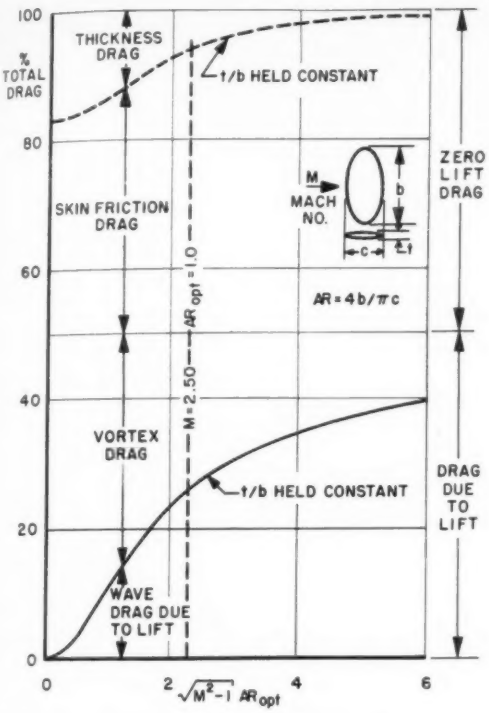

FIG. 6. Percentage composition of drag for optimum elliptic plan-form wings at attitude for maximum ratio of lift to drag versus optimum reduced aspect ratio (from reference 10).

due to lift is approximately evenly divided between vortex and wave drag at Mach 2.5 but becomes predominately wave drag at higher Mach Numbers.

Let us compare these drags at a Mach Number of 2.5 with those customary at high subsonic speeds. From Fig. 5 it is seen that the turbulent skin-friction drag coefficient is some 35 per cent less at this supersonic speed than for a typical jet transport at a Mach Number of 0.83. Skin friction is roughly 88 per cent of the total parasite drag coefficient. Thus, the zero-lift drag coefficient becomes *less* than that at high subsonic speeds by about 26 per cent. At an aspect ratio of unity (which gave the wing the best L/D), the portion of (dC_{D_i}/dC_L^2) due to vortex drag is about *seven* times the subsonic value (corresponding to an aspect ratio of 7.0), and the wave drag coefficient due to lift is of the same magnitude as the vortex drag as seen in Fig. 6. These total *fourteen* times subsonic drag due to lift. The best L/D and the corresponding C_L vary as the square roots of these ratios, Eqs. (5) and (6), (with the values for subscripts 1 and 2 corresponding to the selected airplanes operating at Mach Numbers of 0.83 and 2.5, respectively):

$$\frac{(L/D)_2}{(L/D)_1} = \sqrt{\frac{C_{D_{p_1}}}{C_{D_{p_2}}} \frac{(dC_{D_i}/dC_L^2)_1}{(dC_{D_i}/dC_L^2)_2}} = \sqrt{\frac{1}{0.74} \times \frac{1}{14}} = 0.31 \quad (5)$$

$$\frac{C_{L_2}}{C_{L_1}} = \sqrt{\frac{C_{D_{p_2}}}{C_{D_{p_1}}} \frac{(dC_{D_i}/dC_L^2)_1}{(dC_{D_i}/dC_L^2)_2}} = \sqrt{0.74 \times \frac{1}{14}} = 0.23 \quad (6)$$

The low L/D, 31 per cent of normal subsonic values at a Mach Number of 2.5, results largely from the high value of dC_{D_i}/dC_L^2. Also there is a large reduction in C_L for optimum L/D to about $^1/_4$ normal subsonic values. Cruise altitude and wing loading will be markedly influenced by the low C_L, which tends to maintain these at lower values that the enormous ones that would otherwise be obtained at supersonic speeds by consideration only of the dynamic pressure change due to speed. Reduction of the high dC_{D_i}/dC_L^2 at supersonic speeds is an area of research in which help is greatly needed.

The above analysis has been made for elliptical wings alone. In the build-up of an airplane, tail surfaces will be added having about the same configurations as those of the wing; hence, the L/D of the wing and tail combination will be approximately proportional to that of the wing alone. However, a body for an airplane intended to do a given job generally will have a size principally determined by the mission and load carried rather than the wing area. In a paper presented in 1932 on "Resistance of Slender Bodies Moving with Supersonic Velocities with Special Reference to Projectiles,"[12] Dr. von Kármán (and coauthor Dr. N. B. Moore) presented an ogival nose shape for minimum pressure drag at supersonic speeds. Using this nose shape and assuming that the drag of the total closed body is twice the nose pressure drag plus turbulent skin-friction drag, there have been constructed the total drag coefficients based both on frontal area and (volume)$^{2/3}$ versus fineness ratio as shown in Fig. 7. Bodies of optimum proportions have fineness ratios from about 15 to 25, for which again it is found that the major portion—about $^2/_3$ to $^9/_{10}$—of the drag is skin friction. It appears reasonable then to estimate that at supersonic speeds, as at subsonic

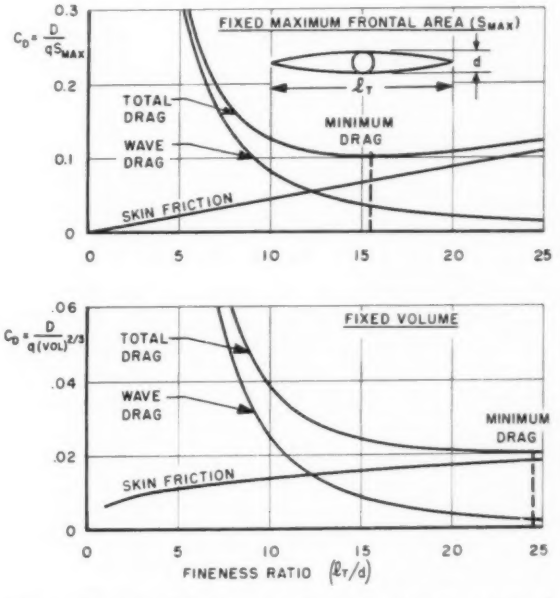

FIG. 7. Drag coefficient versus fineness ratio for bodies of revolution at $M = 2.5$ (assumed $C_f = 0.0018$).

speeds, skin friction can account for half, or possibly a greater portion, of the total drag.

It will be shown in the following discussion that gains in $(L/D)_{max}$ of wings alone can be almost fully transmitted into the same gains in effective $(L/D)_{max}$ of an entire airplane with a given power plant. This point is mentioned since one might be inclined to discount somewhat the importance of wing research in the belief that gains would be largely lost in the airplane as a whole. If the parasite drag other than that of the wing (and tail surfaces) is assumed to be constant, then the total parasite drag coefficient of an airplane can be conveniently written as

$$C_{Dp_A} = \frac{f_A}{S} = C_{Dp_W} + \frac{f_B}{S} = C_{Dp_W}\left(1 + \frac{f_B}{SC_{Dp_W}}\right) \quad (7)$$

where

f_A = drag area of the airplane
= $C_{Dp_A} \times S$ = (total parasite drag coefficient) × (wing area)

C_{Dp_W} = C_{D_p} of wing and tail (assuming tail surface areas proportional to wing area)

f_B = Drag area of the body, nacelles, and other components

Now, if SC_{Dp_W} is kept constant, it can be seen from Eq. (7) that a change in the parasite drag coefficient of the wing (plus tail) will result in a proportional change in the parasite drag coefficient of the entire airplane. The total parasite drag area, f_A, then remains unchanged. An improvement in the coefficient of drag due to lift dC_{D_i}/dC_L^2 of the wing alone is directly reflected in the airplane. Substituting the relations for C_{D_p} and dC_{D_i}/dC_L^2 for the airplane and wing alone into Eq. (5) gives for the lift to drag ratio of the airplane

$$\left(\frac{L}{D}\right)_A = \left(\frac{L}{D}\right)_W \sqrt{\frac{C_{Dp_W}}{C_{Dp_W}\left(1 + \frac{f_B}{SC_{Dp_W}}\right)}} \sqrt{\frac{\left(\frac{dC_{D_i}}{dC_L^2}\right)_W}{\left(\frac{dC_{D_i}}{dC_L^2}\right)_A}} \quad (8)$$

It can be seen that when f_B and SC_{Dp_W} are constant and $(dC_{D_i}/dC_L^2)_A$ is proportional to $(dC_{D_i}/dC_L^2)_W$, then $(L/D)_A$ is proportional to $(L/D)_W$ as follows:

$$\frac{(L/D)_{max_{A_2}}}{(L/D)_{max_{A_1}}} = \frac{(L/D)_{max_{W_2}}}{(L/D)_{max_{W_1}}} \quad (9)$$

Thus, if an improvement can be made in the L/D of the wing alone of an airplane, this improvement can also be made to apply in a large measure to the airplane as a whole.

At a given cruising speed and altitude, the total parasite drag will be unchanged since $D_p = qf_A$ and f_A is kept constant. The drag due to lift at this same cruising condition will be unchanged if weight increases as $(1/C_{Dp_W})^{1/2} \sim S^{1/2} \sim b$ (assuming constant aspect ratio) since total drag due to lift is $D_{i_A} = W_A^2/\pi q b^2$. Hence, with a given power plant, this airplane with an improved wing will have the same altitude and speed performance while carrying a greater weight than the unimproved model and will be operating at an L/D improved in the same ratio as that of the wing alone. An overall improvement in operating L/D has a doubly beneficial effect on range or pay load if ceiling or cruising altitude is limiting. In addition to the direct gain from the better L/D, improved L/D permits carrying a heavier total load at a given altitude; hence, more fuel or pay load. The factors that increase range can be seen by reference to the Breguet equation (Eq. 12, given later).

If, instead of increasing wing area, advantage is taken of improved L/D by reducing engine size, both weight empty and drag—and hence range or pay load—would also improve. It is apparent that benefits of better wing L/D can be imparted to the airplane as a whole in various ways.

Operating Economy at High Speeds

Dr. von Kármán has shown keen interest in aerodynamics applied to the overall efficiency of various methods of propulsion and transportation as can be seen by reference to the paper on "What Price Speed?",[13] the 1950 Thurston Lecture (coauthored with G. Gabrielli). In Chapter VI of his book *Aerodynamics*,[14] published in 1954, he points out "mankind's yearning for speed" and then goes on to discuss "What Price Speed?" concluding generally that "... we have to take into account several points. First, the fundamental economy of the power requirement; second, the practical economy of its use in transportation; third, all the other viewpoints—psychological, political, and the like." It is most gratifying to an aerodynamicist involved in the compromises and limitations of practical designs to realize that Dr. von Kármán not only is interested in, but appreciates the problems, goals, and the means of *practical* transportation, particularly by air.

In connection with the fundamental economy of the power requirement and the practical economy of its use in transportation, it is of interest at this point to take a very brief, though very incomplete, look at the future of transportation by air extended to supersonic speeds.

When viewed through simplified operating cost equations, it is possible to determine the relative importance of various elements that make up the total operating cost. It will be seen that at supersonic speeds with present power plants, the fuel cost becomes predominant, and the total cost can be held to a value little greater than that at high subsonic speeds. Unfortunately, the range at supersonic speeds suffers considerably.

It is customary to evaluate an airplane design for commercial transportation largely by its speed, range, and economy. Economy is usually taken to mean the cost, in appropriate monetary units, of carrying a given unit of load one unit of distance. In American units

for a passenger weight unit of 200 lbs., the usual cost unit, E, is,

$$E = \frac{\text{cents}}{\text{200-lb. nautical mile}} \tag{10}$$

Costs are usually considered in the two categories, direct and indirect. Indirect costs are largely the usual overhead while direct costs include those of crew, maintenance, depreciation, insurance, and fuel. A detailed formula for computing the direct costs has been suggested by the Air Transport Association.[15] This formula is widely used in complete form for route analysis and airplane evaluation.

By combining some terms and discarding others of small magnitude, this ATA equation can be reduced, as shown by Johnstone,* to a very simple formula which enables one to visualize trends and rapidly estimate possibilities in other speed regions. The resulting equation takes the form of the sum of a fixed cost term plus terms involving block speed, air-frame, engine, and fuel costs. In American units, the cost per hour reduces to

$$\text{¢/hr.} = c_o + c_v V_b + c_a U_a W_a + c_e U_e W_e + c_f U_f (W_f/t_b) \tag{11}$$

where

c's = various dimensional constants
U's = unit costs per lb. of item
W's = weights
V_b = block velocity made good, including allowance for maneuver, climb, and acceleration
t_b = block time

subscripts

o = constant, independent of other factors
v = velocity
a = empty airplane less engines
e = engines
f = fuel burned

In this form, Eq. (11) is satisfactory where similar types of power plants are used which vary principally in size alone. However, where the type of power plant differs considerably—and, hence, cost may be expected to be related more reasonably to the thrust output of the engine rather than to its weight—the term for cost per hour due to power plant, $c_e U_e W_e$, might better be replaced by a term, $c_e U_t T_e$, where U_t is now the cost per unit of thrust and T_e is the total rated thrust of the engines.

It is possible to express the last two terms of Eq. (11) in aerodynamic and power-plant parameters. An expression for block range, R_b, follows:

$$R_b = V_b t_b = r_b R_{abs} = r_b H\eta \left(\frac{L}{D}\right) \ln \left(\frac{1}{1 - (W_f/W_G)}\right) = r_b H\eta \left(\frac{L}{D}\right) \frac{W_f}{W_{av}} \text{ (approx.)} \tag{12}$$

where

r_b = ratio of block range to absolute range
R_{abs} = absolute range with design fuel load
H = energy content of fuel (dimensions of length per pound of fuel)
$H\eta$ = $H \times$ (overall power-plant efficiency) = velocity/(thrust specific fuel consumption)
W_G = gross weight
$W_{av} = W_a + W_e + (1/2)\, W_f + W_{PL}$
W_{PL} = weight of pay load

L/D in terms of $(L/D)_{max}$ and k is found from Eq. (1). Introducing Eq. (12) in the last, or fuel cost, term of Eq. (11), there is obtained a relation which will be helpful for studying the influence of aerodynamic design on fuel cost,

$$\frac{W_f}{t_b} = \frac{W_{av} \cdot V_b}{r_b H\eta (L/D)} \text{ (approx.)} \tag{13}$$

The term in Eq. (11) containing W_e is determined by that requirement which is most critical for engine power. This might appropriately be speed, ceiling at cruise speed or with an engine inoperative, take-off distance or climb, etc. For this analysis, only in-flight limitations will be considered. Of these, speed is the most critical. For turbine-powered airplanes flying above the tropopause, it is convenient to use an expression for the ratio of total drag to pressure in terms of parasite drag area alone as follows:

$$D/p = (\gamma/2) M^2 f [1 + (1/k^4)] \tag{14}$$

where

γ = ratio of specific heat at constant pressure to that at constant volume ($\doteq 1.40$ approx.)
p = ambient pressure

In level flight, drag may be replaced by thrust. Then Eq. (14) specifies the engine size needed corresponding to the thrust pressure ratio T/p, which at a chosen Mach Number has approximately a constant value for a given engine above the tropopause (assuming that atmospheric temperature is constant). The required weight of the engine can then be obtained from Eq. (14) by introducing T/p and W_e as follows:

$$W_e = \frac{\gamma}{2} \frac{f M^2}{[(T/p)/W_e]} \left(1 + \frac{1}{k^4}\right) = \frac{\gamma}{2} \frac{f M^2}{Z} \left(1 + \frac{1}{k^4}\right) \tag{15}$$

where:

(T/p) = thrust/pressure ratio at velocity, V, and temperature at altitude
Z = $(T/p)/W_e$

Substituting Eqs. (13) and (15) in the cost equation, we obtain for speed critical designs,

* Unpublished investigations by P. M. Johnstone, Jr., Aerodynamics Section, Engineering Department, Santa Monica Division, Douglas Aircraft Company, Inc.

$$¢/\text{hr.} = c_o + c_v V_b + c_a U_a W_a + c_e U_e \frac{(\gamma/2) f M^2}{Z} \times \left(1 + \frac{1}{k^4}\right) + c_f U_f \frac{W_{av}}{r_b H \eta (L/D)} \frac{V_b}{} \quad (16)$$

The unit of cost is the hourly cost divided by the velocity, V_b, to give per mile cost and then by the number of 200-lb. pay load units, $W_{PL}/200$, to give

$$\frac{\text{cents}}{\text{200-lb. nautical mile}} = 200 \left[\frac{c_o}{W_{PL} V_b} + \frac{c_v}{W_{PL}} + c_a U_a \left(\frac{W_a}{W_{PL}}\right) \frac{1}{V_b} + c_e U_e \frac{\gamma f M^2}{2Z} \left(1 + \frac{1}{k^4}\right) \times \frac{1}{W_{PL} V_b} + c_f U_f \frac{1}{r_b (L/D) H \eta} \frac{W_{av}}{W_{PL}} \frac{1}{V_b} \right] \quad (17)$$

From this equation, the effects of L/D and speed on cost can be readily determined. Now the economy at supersonic speed compared to that at high subsonic speed can be estimated. The ratios of f and $(L/D)_{max}$ previously given for the wing alone are assumed to apply to the complete airplanes. The speed ratio, k, cost factors c and U, component weights, and wing areas are taken to be the same. Since the block velocity enters most of the cost terms, it should be noted that at maximum range the cruise speed ratio of 3:1, for the assumed airplanes cruising at Mach Numbers of 2.5 and 0.83, corresponds to a lower block speed ratio because of allowances for maneuver, climb, and acceleration. The cruise speed ratio of 3:1 is assumed to reduce to a block speed ratio of 2:1 for the chosen airplanes. The relative cost results are shown in Table 1.

Consider the individual terms of Eq. (17) as shown in Table 1 and the changes in each in going to the supersonic transport. For the subsonic jet transport, the first two terms are quite small and the remaining three terms are of nearly equal value. For the supersonic transport, the magnitude of the terms and their dependence on the pertinent variables are the following:

(1) The first term is small and inversely proportional to the block velocity.

(2) The second term is independent of velocity and likewise small in magnitude.

(3) The third term, representing cost due to the airframe, reduces as the block speed increases.

(4) The fourth term, representing costs due to engines, varies inversely with V_b and Z, and directly with f and M^2. Being proportional to fM^2, the requirement for T/p for the supersonic transport is equal to $0.74 \times (2.50/0.83)^2$, or about 6.7 times the value for the subsonic transport. Calculations for turbojet engines without an afterburner and with afterburning show that it is possible to obtain a value of Z at a Mach Number of 2.5 with less than full afterburning that is 6.7 times the value at a Mach Number of 0.83. Fig. 8 shows variations of Z with Mach Number based on computations using typical basic engine component efficiencies. Since engine weight is unchanged, the fourth term changes by the ratio $W_e/V_b = 1.0/2.0$ and becomes 50 per cent of the subsonic value.

(5) The fifth—the fuel cost—term varies inversely with H, η, L/D, and r_b. Fig. 9 shows the variation of overall power-plant efficiency for the turbojet engine without an afterburner and with partial afterburning as needed to obtain the required thrust ratio. The fuel cost term decreases because of about 30 per cent better η, H being constant for one type of fuel. However, the product of $L/D \times \eta$ results in the term increasing in the ratio of $(18/5.6) \times (0.20/0.26) = 2.5$. Correction for an assumed block range factor of 5/6 further increases the cost ratio to $(6/5) \times 2.5 = 3.0$.

The reduction in block range is assumed to be one half that in block speed because time lost at low power does not appreciably influence the range made good. Note that the reduced range as well as the higher speed of the supersonic transport is partially responsible for the large decreases in block speed and range. Significant changes are the marked reduction in costs per 200-lb. nautical mile due to air frame and engine (third and fourth terms) and the large increase in cost due to the fuel (fifth) term, which has become more than 2/3 of the total operating cost. Improvement should be directed toward this item. The total cost per 200-lb. nautical mile has increased from 1.40 cents to 1.77 cents, an increase of 1/4 for the block speed gain of 100 per cent.

If the ratio of pay load to other weights which determine cost can be maintained at supersonic speeds at the usual subsonic values, the current low operating costs could be closely approached up to the lower ranges attainable at supersonic speeds. Many other limitations have to be considered and investigated. However, this rough analysis serves to show a simple means of studying operating costs which indicates good potentialities for successful transportation at supersonic speeds and focuses attention on the places where gains and losses may be expected to occur.

The greatest weakness of the supersonic design as we now visualize it is short range because of the low L/D_{max}, which at present would appear to be limited to about 1/3 of the subsonic values. Improved η partially compensates for this loss, but the block range factor tends to neutralize this gain. On the basis of past progress in range at subsonic speeds, we can have confidence that range will be improved, but many hours of fundamental research and engineering design work will be required.

Before leaving the subject of range and economy, it should be pointed out that in this simplified analysis, wing loading was assumed to be constant and only high altitude in-flight requirements were considered. From the relation for level flight,

$$W/pS = (\gamma/2) M^2 C_L \quad (18)$$

it is seen that cruise altitude for constant wing loading is much higher for the supersonic design than for the subsonic airplane. The ratio of the value of M^2C_L for the supersonic airplane to the value for the subsonic airplane is $3.0^2 \times 0.23 = 2.1$, and since (W/pS) is

TABLE 1

Evaluation of Individual Terms of Operating Cost, Eq. 17, for Assumed Subsonic and Supersonic Jet Transports

TERM NO.	TERM	SUBSONIC TRANSPORT (M = 0.83)		RATIOS OF SUPERSONIC TO SUBSONIC VALUES OF CHANGING FACTORS*	SUPERSONIC TRANSPORT (M = 2.5)	
	200 x :	$ / 200-lb. n. mi.	Percent Total		$ / 200-lb. n. mi.	Percent Total
(1)	$c_o \frac{1}{W_{PL} V_b}$	.06	4	$\frac{1}{V_b} : \frac{1}{2.0} = .50$	.03	2
(2)	$c_v \frac{1}{W_{PL}}$	.04	3	const. : 1.0	.04	2
(3)	$c_a U_a \left(\frac{W_a}{W_{PL}}\right) \frac{1}{V_b}$	.50	36	$\frac{1}{V_b} : \frac{1}{2.0} = .50$	.25	14
(4)	$c_e U_e \frac{W_a}{W_{PL}} \frac{1}{V_b} = c_e U_e \left(1 + \frac{1}{k^4}\right) \frac{\delta f M^2}{2 Z} \frac{1}{W_{PL} V_b}$	.38	27	$\frac{f M^2}{Z V_b} : \frac{.74 \times (3.0)^2}{6.7 \times 2.0} = .50$	.19	11
(5)	$c_f U_f \frac{1}{r_b (L/D) H \eta} \frac{W_{av}}{W_{PL}}$	.42	30	$\frac{1}{r_b (L/D) \eta} : \frac{1}{.83 \times .31 \times 1.3} = 3.0$	1.26	71
	TOTAL	1.40	100		1.77	100

*Ratios of weight factors W_{PL}, W_a, and W_{av} assumed = 1.0.

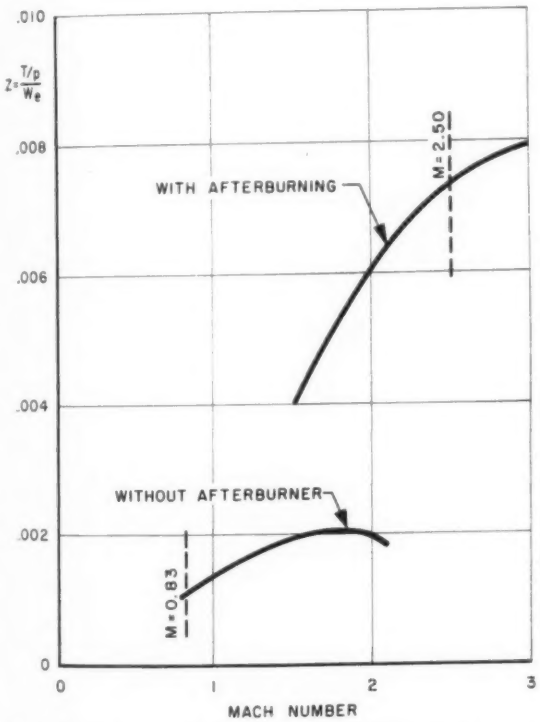

FIG. 8. Variation of Z (ratio of thrust/pressure to engine weight) with Mach Number for a typical turbojet engine without afterburner and with partial afterburning.

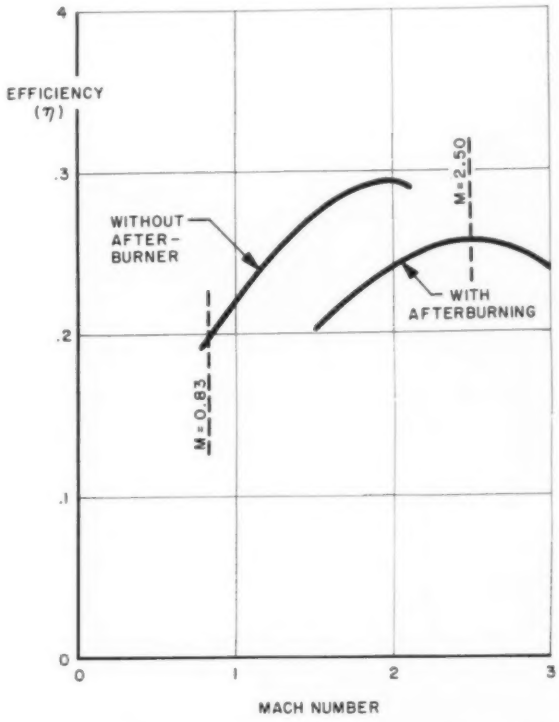

FIG. 9. Variation of overall power-plant efficiency with Mach Number for a typical turbojet engine without afterburner and with partial afterburning.

proportional to M^2C_L, the supersonic airplane will cruise at half the pressure ratio—or at about 50,000 ft.—compared to 35,000 ft. for the subsonic airplane. Should take-off and other critical conditions permit, up to twice the load could be carried in cruising flight without increasing size of engines or airplane at the altitude at which the subsonic speed airplane cruises. For typical fuel weight ratios and structural weights, an increase in range of about 50 per cent would be possible. Thus, we might hope for an improvement up to 1/2 the range of a subsonic transport.

On the other hand, the take-off run and climb at low altitude can be expected to become serious problems. Because of the low L/D of the low aspect ratio wing much added thrust—hence, engine weight—would be needed. Much of the aforementioned possible increase in wing loading would be used up in this manner.

To analyze this low-level requirement would require substitution of the relation $W_e = [W_G/(L/D)]\ (W_e/T_e)$ into the engine cost term. It is apparent that unless the ratio T_e/W is much improved, engine weight, gross weight, and speed would increase markedly. The engine cost term would increase in magnitude. Compromise of the airplane in many respects would have to be made. This more complicated problem is left to those making serious design studies. It is expected that the final conclusions would be much the same as those made from the simple analysis previously presented.

In answer to Dr. von Kármán's question "What Price Speed?" it can be said that the approximate and simplified analysis made herein indicates a possibility of closely approaching, at supersonic speed, present subsonic levels of cost at 2 to 3 times the best subsonic operating speeds but at about 1/3 subsonic range. At present, we lack practical experience in supersonic transport aircraft, but considering past progress as indicative of coming events, economical transportation well into the supersonic speed region can be predicted with assurance.

Ballistic Vehicles and Hypersonic Speeds

In much of his work of recent years, Dr. von Kármán has been concerned with the higher speed regions of flight. He has written extensively on the problems of heating, hypersonic flow, superaerodynamics, and rocket and long-range ballistic flight. Ballistic vehicles are already a part of the work of design aerodynamicists, and there can hardly be any doubt that soon the many problems of flight at hypervelocities must be faced.

One of the most significant changes in airplanes and missiles flying in the supersonic speed region is the enormous increase in the vehicle's energy level. In the future of flight on this planet, it appears that the energy level will change from principally potential to largely kinetic energy. A chart which has been found useful in visualizing this fact and determining some of the requirements of flight at supersonic speed is given in Fig. 10.

By way of introduction to the inherent characteristics of vehicles flying at supersonic and hypersonic

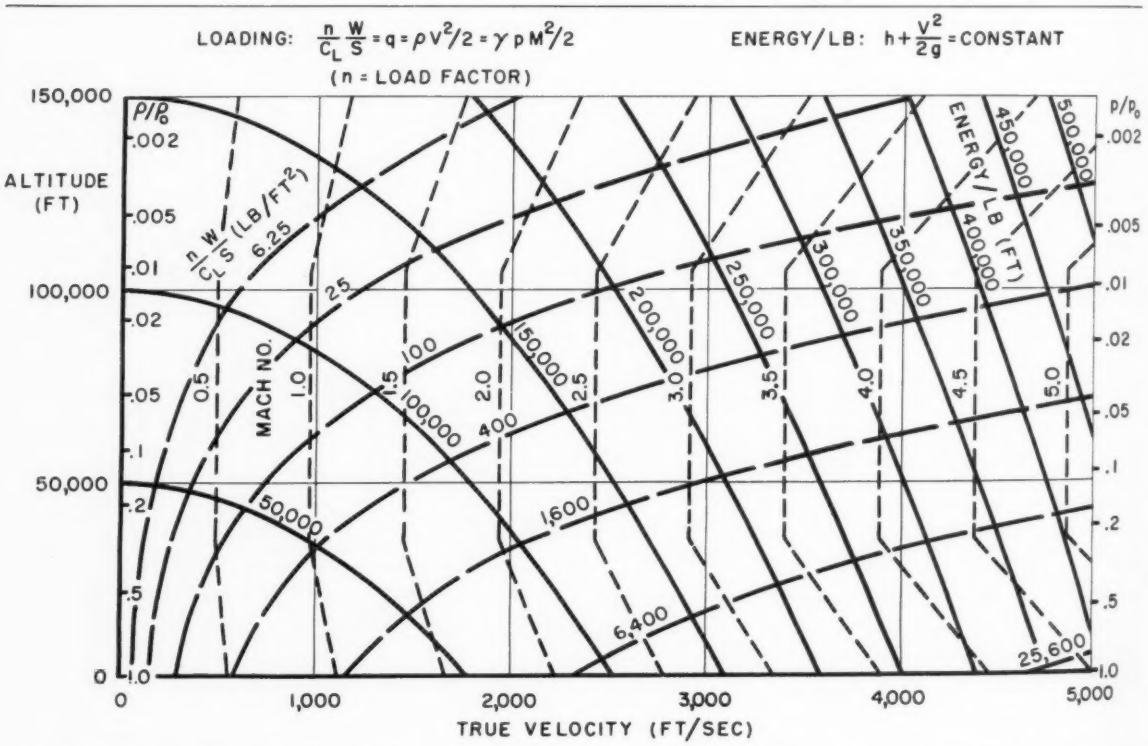

Fig. 10. Speed, altitude, energy, and dynamic pressure (loading) relationships (atmospheric properties from reference 16).

speeds, consider the content and distribution of the energy of aircraft flying at high subsonic and moderate supersonic speeds. In short period maneuvers, a given airplane can make changes in altitude and speed approximately along the curves of constant energy level. Hence in the subsonic speed region, altitude can be readily converted into speed as can be seen from the flatness of the constant energy curves. In contrast, this is not practical at supersonic speeds where at altitudes practical for level flight with air-breathing engines it is more effective to convert a high speed into altitude because of the steepening of the energy curves. Thus, the importance of altitude to flexibility of operation is giving way to speed with its useful kinetic energy. Also note, as can be realized by considering the lines of constant dynamic pressure, $(n/C_L)\ (W/S)$, that level flight at very high altitude will have to be performed at supersonic speeds if reasonable wing loadings are retained.

As speed and altitude increase, factors such as the variation of gravity force with altitude, greater circumferential distance with altitude, centrifugal force, and the weight of the additional air necessary to pressurize the cabin begin to reach significant proportions at high subsonic speeds and above. Surprisingly enough, corrections for these factors, while almost negligible, do change pay load within the accuracy of quoted numbers. Calculations for our assumed jet transport airplane flying at an average weight of 200,000 lbs. and high subsonic speed (Mach Number = 0.83) for which typical pay load, range, and altitude are 30,000 lbs., 3,000 nautical miles, and 35,000 ft., respectively, show the following approximate corrections:

	Effect on Range at Earth's Surface (nautical miles)
(1) Distance increase with altitude (+0.17%) = 5 nautical miles	−5
(2) Gravitational variation with altitude (−0.34%) = −700 lbs.	+10
(3) Centrifugal force effect with speed (−0.10%) = −200 lbs.	+3
(4) Increased weight of cabin air due to pressurization = +500 lbs.	−8

Items 1 through 3 result in about an 8-mile increase in range capability. Consideration of the additional weight of the cabin air almost exactly cancels this net gain. Assuming fuel burned to be half of the total weight, the individual pay load gains are about 2/3 of the total weight changes shown—some being of the order of several hundred pounds.

For our assumed supersonic airplane of the preceding section—flying at a Mach Number of 2.5 at 50,000 ft. altitude and having a range of about 1,000 nautical miles—these factors would change to approximately

	Effect on Range at Earth's Surface (nautical miles)
(1) Distance increase with altitude (+0.24%) = +2.4 nautical miles	−2
(2) Gravitational variation with altitude (−0.48%) = −1,000 lbs.	+5
(3) Centrifugal force effect with speed (−0.88%) = 1,800 lbs.	+9
(4) Increased weight of cabin air due to pressurization = +600 lbs.	−3

In this case, there is a net gain in range ability of 9 miles equivalent to about 1,200 lbs., or 4 per cent of the pay load.

The above discussion serves to show that effects which are generally considered only for ballistic and modified ballistic flight such as glide rockets are in reality already being experienced at the energy levels of today's speeds even if only to a minor degree.

It is well known that the maximum range of a drag-free ballistic rocket of moderate speed (fired at 45° to a plane surface) is proportional to its kinetic energy

$$R = V^2/g_0 \tag{19}$$

where, g_0 is the gravitational acceleration at the earth's surface.

For ground-launched rockets of short range (less than 25 miles), the actual range is approximately 1/2 of the drag-free range. This factor increases as the range increases. More nearly exact relations for the range and other trajectory characteristics of drag-free ballistic rockets optimally launched to give maximum range with an impulsive velocity V_0 at an angle δ_{0max} have been derived in useful form by Tschirgi.* The gravitational force is assumed to be centrally directed and to vary inversely with the square of the distance from the center of the earth. Rotation of the earth is not accounted for, but correction may be made roughly by consideration of resultant angles and velocity vectors and the net movement at the earth's surface due to its rotation during the time of flight. It should be noted that after such correction, the trajectories no longer would be optimum.

For these optimum trajectories, the various expressions for range R_{max}, launching angle δ_{0max}, maximum altitude h_{max}, and time of flight t_{fmax} follow:

(a) Range

$$R_{max} = \left[\pi - 2 \arcsin \sqrt{\frac{1 - (V_0^2/g_0R_0)}{1 - (V_0^2/2g_0R_0)}}\right] R_0 = \theta\, R_0 \tag{20}$$

* Unpublished investigations by J. M. Tschirgi, Aerodynamics Section, Engineering Department, Santa Monica Division, Douglas Aircraft Company, Inc.

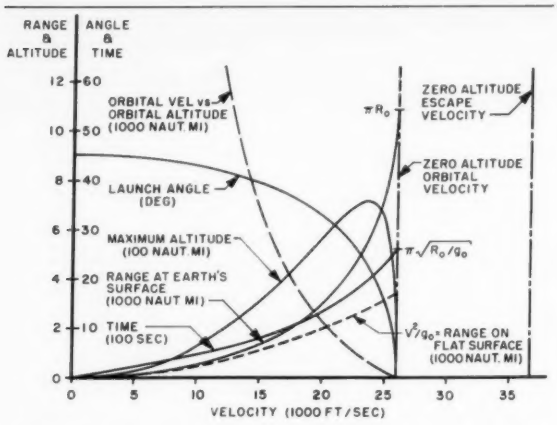
RANGE & ALTITUDE
ANGLE & TIME
ORBITAL VEL vs ORBITAL ALTITUDE (1000 NAUT MI)
LAUNCH ANGLE (DEG)
MAXIMUM ALTITUDE (100 NAUT MI)
RANGE AT EARTH'S SURFACE (1000 NAUT MI)
TIME (100 SEC)
ZERO ALTITUDE ESCAPE VELOCITY
ZERO ALTITUDE ORBITAL VELOCITY
V²/g₀ = RANGE ON FLAT SURFACE (1000 NAUT MI)
VELOCITY (1000 FT/SEC)

FIG. 11. Summary of optimum drag-free ballistic trajectory characteristics about the earth as functions of impulsive velocity assuming no atmosphere and nonrotating earth.

where

θ = angle subtended at the center of the earth between points of launching and landing (radians)

R_0 = radius of earth

(b) Launching angle

$$\delta_{0max} = \arccos \sqrt{\frac{1}{2 - (V_0^2/g_0R_0)}} \tag{21}$$

(c) Maximum altitude

$$h_{max} = (1/2)\,[\sin 2\,\delta_{0max} + \cos 2\,\delta_{0max} - 1]\,R_0 \tag{22}$$

(d) Time of flight

$$t_{fmax} = \frac{2\sqrt{R_0/g_0}}{\sqrt{[2 - (V_0^2/g_0R)]^3}} \times \left[\sqrt{\frac{V_0^2}{g_0R_0} - \left(\frac{V_0^2}{g_0R_0}\right)^2} + \arcsin \sqrt{\frac{V_0^2}{g_0R_0}}\right] \tag{23}$$

Fig. 11 presents the variation of range, launch angle, maximum altitude, time, and kinetic energy per unit weight (corresponding to range on a flat surface) with impulsive velocity for optimally launched drag-free rockets.

Impulsive velocity of a drag-free rocket is obtained from the specific impulse, i_{sp}, and the propellant and gross weights, W_f and W_G, respectively, by the relation

$$V_0 = g_0\, i_{sp} \ln\left[\frac{1}{1 - (W_f/W_G)}\right] \tag{24}$$

where

i_{sp} (sec.) = impulse (lb.-sec.) per lb. of propellant

It is assumed in Eq. (24) that the burning time is negligibly small in order to establish the full velocity instantaneously. The influence of gravity during burning can, therefore, be neglected. The effect of staging is given by adding the velocities corresponding to the weight ratios for each stage, which is equivalent to taking the logarithm of the product of the terms in parentheses for each stage.

Eq. (20) simplifies to Eq. (19) at moderate speeds. From Eq. (20) it is seen that, although the range of a ballistic rocket is primarily a function of its energy content, there no longer exists a simple proportionality between kinetic energy and range as the velocity is increased. Whereas the approximate result for short range (Eq. 19) shows that the range increases as the square of the launching velocity, the more nearly exact result for a spherical earth shows that the range increases with velocity faster than the square. It becomes infinite at the theoretical zero altitude orbital (satellite) velocity of about 25,900 ft./sec.

Also shown in Fig. 11 is the variation of orbital velocity with orbital altitude, h, obtained from the expression

$$V_{orbital} = R_0 \sqrt{g_0/(R_0 + h)} \tag{25}$$

Remarks concerning the relations between long range and high kinetic energy content apply equally to either ballistic or glide vehicles. High speed and long range are much desired, and they are simultaneously obtained by flying at hypersonic speeds.

It is noteworthy that the ability to turn is rapidly being lost at high supersonic speeds. Operations may well be restricted to one-way trips with direction established early in the flight program.

In his book *Aerodynamics*,[14] Dr. von Kármán points out that there are problems in reaching satellite speed, which, with present chemical fuels, would be solved only with staging. The initial velocity needed for escape of a vehicle from the earth's gravitational field is 36,600 ft./sec. and is given by the expression

$$V_{escape} = \sqrt{2g_0\,R_0} = \sqrt{2}\;V_{satellite} \tag{26}$$

Reaching escape velocity is still more difficult and may require some application of nuclear power. Dr. von Kármán further suggests that "the next decade probably will show what can be accomplished by the use of nuclear reaction in the field of jet propulsion."

Lest we feel that with such flight accomplishment near our Earth, achievement of high speeds will finally end, let us consider what might be done well within the physiological capacity of a human being, the propulsion device permitting. Let it be assumed that a net acceleration away from the Earth of $1g_0$ is applied to a vehicle for about 18 hours. At the Earth's surface, the force felt by a person in this vehicle corresponds to $2g_0$—that is, twice normal weight. In about 20 min. the vehicle would have proceeded far enough from the Earth's surface so that this force would be reduced to a value corresponding to $(5/4)g_0$ and soon thereafter to $1g_0$—at which, of course, man is accustomed to existing. Experiments today indicate that it would be possible for a man to withstand these moderately higher accelerations for the times assumed. If, at the end of the 18 hours of applied $1g_0$ acceleration, the approximate time history of the acceleration is reversed, the vehicle could then come to rest on the

planet Venus, 22 million nautical miles distant from the Earth, 36 hours after take-off. The speed of the vehicle would have reached 1,200,000 nautical miles per hour or about 350 nautical miles per sec. This speed is about 2,000 times the speed of sound in the terrestrial stratosphere, or about 1/500 of the speed of light (161,000 nautical miles per sec.). At its maximum velocity, the vehicle would be traveling at about 20 times the speed of the Earth (or Venus); hence, it would be in a commanding situation for interplanetary travel.

In Section A of *General Theory of High Speed Aerodynamics* entitled "On the Foundation of High Speed Aerodynamics,"[8] Dr. von Kármán, with characteristic perception of future problems of importance, reviewed the fundamentals of flight problems at hypersonic speeds. He discussed the inviscid flow of air about thin wings and slender bodies and pointed out that the properties of the flow in the hypersonic speed range are basically different from those in the familiar linearized supersonic speed range. In spite of the fact that the linearized supersonic theory is highly simplified and idealized, it does provide a very useful engineering approximation for the analysis of thin wings and bodies at small angles of attack. In the linearized supersonic speed range, the Mach angle μ ($\sin \mu = 1/M$) is large compared with the maximum slope of the body. In the hypersonic speed range, the Mach angle is of the order of or less than the maximum slope of the body. In linearized supersonic flow, the shock waves are straight and of infinitesimal strength. The shock angle is equal to the Mach angle μ and independent of the body shape. At hypersonic speeds these simplifications cannot be made. At hypersonic speeds, the differential equations that govern the flow about slender configurations are necessarily nonlinear, and it appears that they cannot in general be solved analytically without other approximations. To further illustrate the difference, consider the compression side of a two-dimensional flat-plate airfoil at a small angle of attack α. At supersonic speeds, the pressure coefficient is proportional to α and, for a given α, decreases with increasing Mach Number. At very high hypersonic Mach Numbers, the pressure coefficient is proportional to α^2 and is independent of Mach Number; it is interesting to note that for this case the flow conditions are very similar to the picture visualized by Newton—i.e., the air particles hit the body and continue their paths along the surface. The design engineer is just beginning to face problems in the hypersonic speed range.

As we move into higher supersonic and hypersonic speeds, the greatest problem besetting the designer is heating in the boundary layer. In early supersonic design work, viscous phenomena were largely neglected in a preoccupation with the problems of shock-induced pressure drag. The pendulum has now swung to the opposite side, and it is known that the most serious problem pertaining to hypervelocity flight within the atmosphere is boundary-layer heating. The burning of a meteor upon entering the earth's atmosphere is indicative of the seriousness of the problem. As a pioneer in the field of viscous flows, the heating problem has been considered by Dr. von Kármán in developing the turbulent skin-friction relation at supersonic speeds. He has pointed out the effects of this heating on the boundary-layer flow and structure located in this region. Laminar flow would largely reduce skin-friction heating, and there is some possibility of the natural attainment of this smooth flow at quite high supersonic speeds and altitudes. In any event, the designer involved in the design of aircraft flying above a Mach Number of 3 and of ballistic reentry vehicles will be much concerned with structures made of materials having special properties at high temperature and with means of cooling the surface internally as well as externally.

Not content with normal problems of today, Dr. von Kármán has delved into analysis of the slip flow region This region is still largely untried and untested. Indicative of Dr. von Kármán's keen interest in these advanced fields is his following observation on superaerodynamics and the slip-flow region[8]: " . . . the characteristic feature of the slip flow is that the viscous stress is no longer small in comparison with the ambient pressure. On the other hand, the transonic and supersonic speed range is characterized by the fact that the momentum per unit area and unit time, ρU^2, is not small in comparison with the ambient pressure, since evidently $\rho U^2/p = \gamma M^2$. In the most interesting problems of superaerodynamics, both M^2 and M^2/Re^* are large. In other words, both the momentum ρU^2 [or the dynamic pressure $(1/2)\rho U^2$] and the viscous stresses are large in comparison with ambient pressure." It is gratifying to know that Dr. von Kármán is directing his thoughts to the problems of these hypersonic speeds. Again we can confidently expect that practical flight in this regime of speed and altitude will be effectively accomplished.

Conclusion

To the design aerodynamicist, who must be concerned with overall aerodynamic design in its many aspects including those related to other fields, Dr. von Kármán's contributions to the progress of aviation have been most apparent. He has presented theories, methods, trends, and analyses of use in design, appearing to direct his efforts largely to those problems most in need of solution such as turbulent skin friction, compressibility effects, supersonic drag of various kinds, hypersonics, superaerodynamics, and operating efficiency.

The importance of his contributions on skin-friction drag can hardly be overemphasized. This type of resistance now comprises the greater portion of parasite drag at subsonic speeds, and it will continue to do so at supersonic speeds. Turbulent skin friction now accounts for about 75 per cent of the parasite drag

* Re = Reynolds Number

and 50 per cent of the total drag of an airplane in cruising flight. There should be little, if any, reduction in these proportions at supersonic speeds. The predominance of skin friction will result from the use of extremely thin low aspect ratio wings. A design study of unswept elliptical wings taking account of structural as well as aerodynamic factors has shown very low aspect ratio wings to be most efficient. Improvements in lift to drag ratio of a wing can likewise be reflected in similar improvements of the airplane as a whole.

The maximum lift to drag ratio at supersonic speeds currently appears to be limited to about $^1/_3$ the normal subsonic values. This low $(L/D)_{max}$ results from a manifold increase in drag due to lift, not from increased zero-lift drag. The drag at zero lift should remain equal to or less than subsonic values as a result of the reduction in skin friction at supersonic speeds, in accordance with the trend shown by Dr. von Kármán's theory.

The lower lift to drag ratio, although partially compensated for by better overall power-plant efficiency of turbojet engines at supersonic speeds, leads to much lower range for a supersonic transport. The past record would indicate that marked improvement in range can be expected with continuation of serious efforts in aeronautical research and design. Until considerable improvement in range can be made, the supersonic transport would be subject to the usual complications encountered when operating an airplane over routes requiring important flights longer than the maximum range of the airplane.

By an analysis using greatly simplified operating cost equations, it has been shown that up to the range attainable, the operating economy at supersonic speeds may be kept close to present subsonic values. This result is largely due to high speed and to the high output and overall efficiency of the turbojet engines with afterburning. The low $(L/D)_{max}$ at supersonic speed causes the fuel cost term to become predominant, and the higher speed causes the other important cost terms to be markedly reduced.

Speeds are approaching values where the kinetic energy is becoming predominant and especially significant. Since range is obtained ballistically from speed, flight into the hypersonic speed region, where the pressure relations tend to follow more closely the early Newtonian idea, is readily envisioned. For space flight, possibly with nuclear propulsion, it is anticipated that speed may reach an appreciable fraction of the speed of light. Dr. von Kármán recently has been concerned with problems of heating and the slip-flow region (superaerodynamics of rarefied air). His work here, as in other speed domains, is certain to be of great use to the practicing aerodynamicist who, most assuredly, will be designing vehicles to fly in these regions of very high altitude and speed.

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Heat Transfer in Turbulent Shear Flow

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SUMMARY

The problems of heat transfer in turbulent shear flow along a smooth wall are discussed from the point of view of von Kármán's well-known 1939 paper on the analogy between fluid friction and heat transfer. Methods for extending the analysis to higher Prandtl Numbers are suggested.

SYMBOLS

B = constant in turbulent core velocity profile
$C_f = 2\tau_0/\rho u_m^2$ = friction coefficient
$C_h = q_0/\rho c_p U_m(T_w - T_m)$ = heat-transfer coefficient
c_p = specific heat at constant pressure
D = pipe diameter
k = thermal conductivity
K = constant in turbulent core velocity profile
K_1 = constant in wall layer profile
$Nu = q_0 D/(T_w - T_m)k$ = Nusselt number
q = heat-transfer rate per unit area in y-direction, q_0 the value at the wall
$Re = Du_m/\nu$ = Reynolds Number
T = local mean temperature, T_w at the wall, T_m pipe mean
u = local mean velocity along wall, u_m pipe mean
$u_\tau = \sqrt{\tau_0/\rho}$ = friction velocity
y = coordinate distance from wall
$y^* = yu_\tau/\nu$ = dimensionless distance from wall, y_1^* to edge of turbulent core, y_2^* to outermost distance of appreciable molecular transport of heat or momentum
ϵ = eddy viscosity
$\kappa = k/\rho c_p$ = thermal diffusivity
μ = viscosity
$\nu = \mu/\rho$ = kinematic viscosity
ρ = fluid density
$\sigma = \mu c_p/k$ = Prandtl Number
τ = shearing stress, τ_0 at the wall

INTRODUCTION

THE FIRST IMPORTANT STEP in the understanding of heat transfer in turbulent flow was made by Reynolds[1] in 1874 when he postulated the analogy between heat and momentum transfer in turbulent shear flow. For turbulent shear flow parallel to the x-axis, with velocity and temperature gradients predominant in the y-direction, the analogy can be expressed as

$$\frac{-q}{\rho C_p dT/dy} = \frac{\tau}{\rho du/dy} \qquad (1)$$

where q is the rate of heat transfer per unit area in the y-direction, and τ is the shearing stress. For laminar shear flow, $q = -k dT/dy$ and $\tau = \mu du/dy$, and one sees that the expression (1) above holds if, and only if, the Prandtl Number σ is unity. Hence the Reynolds analogy is strictly correct for laminar shear flow provided the fluid properties are such that $\sigma = 1$. The Prandtl Numbers of most gases are close to unity ($\sigma \cong 0.75$), so the Reynolds analogy is approximately correct up to the wall even though the flow is laminar close to the wall.

The first extensions of the application of Reynolds analogy to fluids with Prandtl Numbers differing from unity were made by Taylor[2] in 1919 and Prandtl[3] in 1926. The extensions consisted essentially of dividing the flow into two layers: a laminar layer next to the wall where the transport processes were assumed entirely molecular, and a turbulent region beyond where the molecular transport processes were neglected. Although these methods gave an improvement over the application of the Reynolds postulate for Prandtl Numbers differing moderately from unity, they suffered from the rather arbitrary choice of the thickness of the laminar layer.

In 1934, von Kármán[4] made a suggestion for improving the heat-transfer theory of fluids with Prandtl Numbers differing from unity, and, in 1939, he carried out his own suggestion in detail. This second paper, "The Analogy Between Fluid Friction and Heat Transfer," was published in the Transactions of the American Society of Mechanical Engineers and has become a classic to those working in the field of heat transfer. The significant step made by von Kármán was to introduce a buffer layer between a laminar sublayer next to the wall and the turbulent core. In the thin laminar sublayer, the transport processes were assumed purely molecular; in the turbulent core, the Reynolds analogy was used; while in the buffer layer, both molecular and turbulent transports were assumed, the first decreasing and the second increasing with distance from the wall. The Reynolds analogy was applied to the turbulent components of the transport terms within the buffer layer.

The importance of von Kármán's contribution consisted largely of the very clear demonstration of the essential transport phenomena. He confirmed the results of his analysis by comparison with heat-transfer measurements for water. The particular choice of laminar sublayer thickness was intended for a range of Prandtl Numbers moderately larger than unity. It is rather surprising that the concepts introduced by von Kármán have had, since that time, so little extension to liquids with higher Prandtl Numbers. Some of the problems in such extension are discussed below.

MOMENTUM AND ENERGY TRANSFER

Consider plane shear flow or the flow in a pipe far from the center; the momentum and energy equations,

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averaged over the turbulent fluctuations, become

$$\tau_0 = -\rho\overline{u'v'} + \mu(du/dy) \tag{2}$$

$$-q_0 = -\rho c_p\overline{T'v'} + k(dT/dy) \tag{3}$$

The total shear stress τ_0, assumed constant, is the sum of the Reynolds stress $-\rho\overline{u'v'}$ and the laminar stress $\mu(du/dy)$; and the rate of heat transfer q_0, also assumed constant, is the sum of analogous components. It is convenient to introduce an eddy viscosity ϵ and to write Eq. (2) as

$$\tau_0/\rho = (\epsilon + \nu)(du/dy) \tag{4}$$

where ϵ is the turbulent counterpart of the kinematic viscosity ν. The latter is, of course, a property of the fluid, whereas ϵ varies with y and must vanish at $y = 0$ for a smooth wall. The energy transport equation may be written in a similar form

$$-(q_0/\rho c_p) = (\epsilon_T + \kappa)(dT/dy) \tag{5}$$

where ϵ_T is the eddy diffusivity.

Reynolds analogy is equivalent to the assumption that $\epsilon_T = \epsilon$ for flows where the molecular components are negligible. Following von Kármán, the same assumption will be used even when ν and κ are not negligible compared with ϵ. The Reynolds analogy does not hold in all types of shear flow. It is well known that the turbulent temperature wake behind a heated body spreads faster than the velocity wake; for instance, Corrsin[5] has found a "turbulent Prandtl Number" $\epsilon/\epsilon_T \cong 0.75$ in a particular example. This is to be expected in a jet or wake far downstream where the u' and v' velocity components are of comparable magnitude, because the momentum transport involves transport of a component of a vector and energy involves the transport of a scalar. Close to a wall the flow is confined strongly to planes parallel to the wall, and it seems likely that the transport of the u component of momentum is similar to transport of a scalar quantity. Measurements of heat transfer in pipes for gases, where the Prandtl Number correction is small, support the Reynolds analogy for shear flow near a wall.

Returning to Eqs. (4) and (5) and introducing the friction velocity $u_\tau = \sqrt{\tau_0/\rho}$ and a dimensionless distance from the wall $y^* = yu_\tau/\nu$, the equations can be rewritten as

$$\frac{d(u/u_\tau)}{dy^*} = \frac{1}{(\epsilon/\nu) + 1}$$

$$-\frac{\rho c_p u_\tau}{q_0}\frac{dT}{dy^*} = \frac{1}{(\epsilon/\nu) + (1/\sigma)}$$

These equations can be integrated in principle to give the velocity and temperature profiles; these are

$$\frac{u}{u_\tau} = \int_0^{y_2^*} \frac{dy^*}{(\epsilon/\nu) + 1} + \int_{y_2^*}^{y^*} \frac{dy^*}{\epsilon/\nu}$$

$$\frac{\rho c_p u_\tau}{q_0}(T_w - T) = \int_0^{y_2^*} \frac{dy^*}{(\epsilon/\nu) + (1/\sigma)} + \int_{y_2^*}^{y^*} \frac{dy^*}{\epsilon/\nu}$$

where y_2^* is sufficiently large that 1 or $1/\sigma$, whichever is larger, becomes negligible compared with ϵ/ν. If $\sigma \ll 1$, the formulation above may not be satisfactory, and modifications similar to those used by Martinelli[6] can be introduced.

The upper limit y^* for the second integral in the expression for u/u_τ can always be chosen so $u = u_m$, the average velocity, as for a pipe. Then, with a small error, the temperature evaluated at the same y^* is the average fluid temperature T_m (see Appendix for a more accurate evaluation). With these limits for the integrals, the equations above are subtracted to give

$$(\rho c_p u_\tau/q_0)(T_w - T_m) - (u_m/u_\tau) = F(\sigma) \tag{6}$$

where

$$F(\sigma) = \int_0^{y_2^*}\left[\frac{1}{(\epsilon/\nu) + (1/\sigma)} - \frac{1}{(\epsilon/\nu) + 1}\right]dy^* \tag{7}$$

The friction coefficient C_f and the heat-transfer coefficient (Stanton number) C_h are defined by the relations

$$C_f/2 = \tau_0/(\rho u_m^2)$$

$$C_h = q_0/[\rho c_p u_m(T_w - T_m)]$$

Multiplying Eq. (6) through by u_m/u_τ, the equation relating heat transfer and friction is

$$1/C_h = (2/C_f) + F(\sigma)\sqrt{2/C_f} \tag{8}$$

where $F(\sigma)$ is given by Eq. (7) above. For $\sigma = 1$, $F(\sigma) = 0$, and the Reynolds analogy holds in both laminar and turbulent regions giving $C_h = C_f/2$.

The Distribution of Eddy Viscosity

For Prandtl Numbers differing from unity, the relation between the heat-transfer and friction coefficients depends on the distribution of eddy viscosity ϵ between $y^* = 0$ and $y^* = y_2^*$. The most direct way of determining ϵ is from the velocity profile since

$$\frac{\epsilon}{\nu} = \frac{1}{d(u/u_\tau)/dy^*} - 1$$

For $y^* \gtrsim 30$—i.e., in the fully turbulent region—the von Kármán logarithmic formula can be used; this was given in the form

$$u/u_\tau = (1/\sqrt{K})\ln y^* + B \tag{9}$$

and the constants $1/\sqrt{K} \cong 2.5$, $B = 5.5$ are well established by experiments on pipes. Hence

$$\epsilon/\nu = \sqrt{K}\,y^* - 1 \qquad y^* \gtrsim 30 \tag{10}$$

For the range $0 < y^* < 30$, von Kármán approximated the measured velocity profile by $u/u_\tau = y^*$ in the laminar sublayer $0 < y^* < 5$, and by a relation of the form of Eq. (9), with different constants, in the buffer layer $5 < y^* < 30$. Then $\epsilon/\nu = 0$ for $y^* < 5$, and this is a satisfactory approximation for moderately large Prandtl Numbers.

There is no reason for believing that ϵ/ν should vanish completely except at the wall itself. The velocity profile near the wall would be virtually unaffected by

an error in the very small value of ϵ/ν in that region. However, an examination of Eq. (7) shows that for $\sigma \gg 1$, the very small values of ϵ/ν close to the wall can have a profound influence on the value of the first integral. There is no possibility of determining ϵ/ν accurately for $y^* < 5$ from measurements because the velocity gradient is required, and it is impractical to attempt to find these from the measured profile. Even the remarkably good experiments of Reichardt,[7] Laufer,[8] and Skinner[9] are of no help in this regard.

Since it does not seem possible to find heat-transfer coefficients for high Prandtl Numbers from measured velocity profiles, it is natural to examine the possibility of finding the velocity profile from heat-transfer measurements. In principle, this is attractive because temperature differences, flow rates, and pressure drop can be measured quite accurately. By heat-transfer measurements on fluids of different Prandtl Numbers, $F(\sigma)$ could be determined as a function of σ, and Eq. (7) then is an integral equation for ϵ/ν. The difficulty with heat-transfer measurements in the past has been that the physical properties, particularly the viscosity, are sensitive functions of temperature for nonmetallic liquids. In order to have an appreciable amount of heat transferred from the wall, it is necessary to use a large temperature difference between the wall and the bulk of the fluid. Frequently, the viscosity changes by a factor of two or three or more from the wall to the edge of the turbulent core. Some attempt is usually made to allow for the viscosity variation by various averaging processes. Reported heat-transfer measurements often show "scatter" of ±20 per cent; it seems likely that the spread is not so much an indication of experimental error as a gross oversimplification of the influence of variation of viscosity. Careful experiments of the type made by Eagle and Ferguson[10] on water, with extrapolation to zero temperature differences, are needed for liquids with higher Prandtl Numbers.

Proposed Velocity Profile

In the absence of sufficiently precise heat-transfer measurements, the writer attempted to find a reasonable velocity profile for the region close to the wall, so the eddy viscosity determined from the velocity led to satisfactory variation of heat transfer with Prandtl Number. The simplest analytical expression that fitted the required conditions was of the form

$$u/u_\tau = (1/\sqrt{K_1}) \tanh (\sqrt{K_1}\, y^*) \qquad (11)$$

This profile was joined to the logarithmic profile, Eq. (9), at a distance y_1^*, so the velocity and velocity gradient were continuous at the junction. The constant K_1 and the distance y_1^* were determined from these two conditions giving

$$y_1^* = 27.5 \qquad 1/\sqrt{K_1} = 14.53 \qquad (12)$$

Comparison of the velocity profile, Eq. (11), with experimental results gives quite satisfactory agreement; of course many other analytical forms for the profile would be equally satisfactory, but this one is very convenient.

With the velocity profile as given by Eq. (11), the corresponding eddy viscosity is

$$\epsilon/\nu = \sinh^2(\sqrt{K_1}\, y^*) \qquad (13)$$

and the function $F(\sigma)$ can be evaluated in closed form.

$$\left.\begin{aligned} F(\sigma) &= \frac{\sigma}{\sqrt{K_1}\sqrt{\sigma-1}} \tan^{-1}\left(\sqrt{\sigma-1}\sqrt{K_1}\, y_1^*\right) - B - \frac{1}{\sqrt{K}} \ln\left(y_1^* - \frac{1}{2\sqrt{K}} + \frac{1}{\sqrt{K\sigma}}\right) \quad \sigma > 1 \\ &= \frac{\sigma}{\sqrt{K_1}\sqrt{1-\sigma}} \tanh^{-1}\left(\sqrt{1-\sigma}\sqrt{K_1}\, y_1^*\right) - B - \frac{1}{\sqrt{K}} \ln\left(y_1^* - \frac{1}{2\sqrt{K}} + \frac{1}{\sqrt{K\sigma}}\right) \quad \sigma < 1 \end{aligned}\right\} \qquad (14)$$

The asymptotic form for $F(\sigma)$ as σ becomes large is

$$F(\sigma) \to 22.8\sqrt{\sigma} - 29.1 \qquad \text{as } \sigma \to \infty \qquad (15)$$

and this is only 5 per cent smaller than the correct value at $\sigma = 10$.

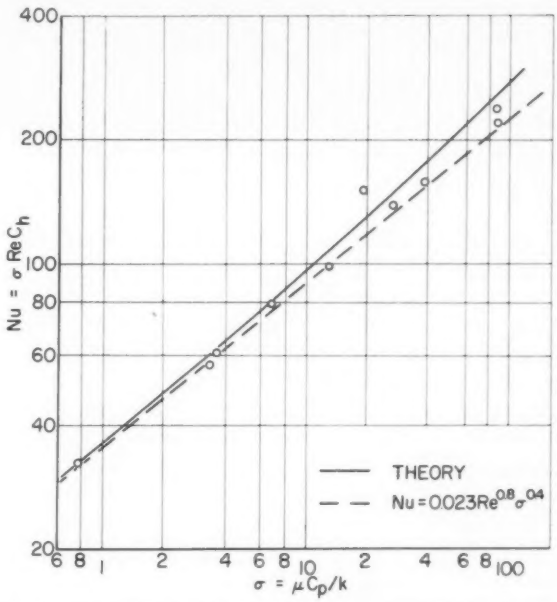

FIG. 1. Dependence of Nusselt number on Prandtl Number at $Re = 10^4$. (Experimental data from Sherwood.)

In order to show that the value of $F(\sigma)$ is in agreement with experiment, a comparison is made for the Prandtl Number range between 1 and 100 in Fig. 1. The experimental points are those of Sherwood as reported by McAdams.[11] The friction coefficient was evaluated

from the empirical relation $C_f = 0.046/Re^{0.2}$. As can be seen, the fit is reasonably good; it must be remembered that the variations in viscosity that occurred near the wall in the experiments are ignored. Also shown is one of the well-known empirical formulas[12] for the range $1 < \sigma < 120$.

$$C_h = 0.023(Re)^{-0.2}\sigma^{-0.6} \tag{16}$$

The theory above implies that $C_h \propto \sigma^{-0.5}$ as σ becomes very large. Few reliable measurements of heat transfer at very high Prandtl Numbers ($\sigma > 500$) have been made, but those that are available[13] seem to indicate that $C_h \propto \sigma^{-2/3}$ for σ very large. This result, if correct, would require that $\epsilon/\nu \propto y^{*3}$ for y^* very small. According to Eq. (13), the turbulent and molecular transports of heat are equal at a value of y^* given by $\sinh \sqrt{K_1}\, y_0^* = 1/\sqrt{\sigma}$; hence at $\sigma = 100$ $y_0^* \cong 1.45$. Then one would expect that the velocity profile, Eq. (11), describes the actual velocity reasonably well for $y^* \gtrsim 1.5$ but not necessarily for smaller values of y^*, unless the agreement indicated in Fig. 1 can be confirmed at higher values of the Prandtl Number. Many other forms for the velocity profile that have been proposed—for instance that of Deissler[14]—are not consistent with heat-transfer measurements at the higher Prandtl Numbers.

Fluids with Variable Physical Properties

As was mentioned previously, heat-transfer measurements in liquids with Prandtl Numbers larger than unity usually involve variable viscosity. If for no other reason than interpretation of experiments, an understanding of the influence of variable physical properties on friction and heat transfer is desirable. An important simplification resulting from high Prandtl Numbers is that most of the temperature difference occurs in the layer between the wall and $y^* = 30$, so the turbulent core is essentially isothermal.

Measurements of Laufer[8] and others of the magnitude of the velocity fluctuations near a smooth wall show that the maximum values occur at $y^* \cong 25$ or 30 and decrease rapidly to zero at the wall. It seems probable that the wall layer within these limits is inherently stable and that the velocity fluctuations are responses of a highly damped layer to disturbances arising outside. The fluctuations inside the wall layer would then have no direct influence on the outside flow.

If the temperature variation is confined to the damped layer of fluid next to the wall, the viscosity variation is confined to this region as well. The turbulent core is isothermal and can be considered as having known velocity gradient distribution and as producing known disturbances at the edge of the wall layer. The response at any point of the wall layer can then in principle be determined, even if the viscosity varies strongly through the layer. The turbulent shear stress $-\rho\overline{u'v'}$ will depend on the viscosity variation, but, with the model above, there is some hope of estimating this dependence. The writer[15] has attempted to use such a model for the wall layer to determine ϵ/ν as a function of the viscosity variation and hence to find the friction coefficient and the heat-transfer coefficient. Although the results are promising, confirmation will not be possible until more reliable heat-transfer and friction measurements are made for liquids at high Prandtl Numbers.

In gases, about half the temperature difference between the wall and the mean temperature occurs in the wall layer so that the turbulent core is not isothermal. Then the disturbances of the damped wall layer cannot be considered as known because the turbulent core cannot be identified with a similar one in isothermal flow. Further, the density fluctuations in gases may introduce other complications not found in liquids. An understanding, or even a reliable description of the turbulent transport processes close to a wall for high rates of heat transfer, seems much farther in the future for gases than for liquids at high Prandtl Numbers.

Appendix

A more satisfactory derivation of Eq. (8) can be carried out by defining mean velocity and mean temperature for pipe flow with appropriate weighting factors—e.g.,

$$\frac{u_m}{u_\tau} = \frac{2}{R^{*2}}\int_0^{R^*} \frac{u}{u_\tau}(R^* - y^*)dy^*$$

$$T_w - T_m = \frac{2}{R^{*2}(u_m/u_\tau)}\int_0^{R^*} (T_w - T)\frac{u}{u_\tau} \times (R^* - y^*)dy^*$$

where $R^* = Ru_\tau/\nu$ and R is the pipe radius. If the evaluation of the integrands is made with terms of order $1/R^*$ and higher neglected, Eq. (8) is replaced by

$$1/C_h = (2/C_f) + F(\sigma)\sqrt{2/C_f} + 5/4K$$

The additional term on the right, approximately 8.8, is an unimportant correction for most applications.

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Applied Mathematics

(*Continued from page 410*)

tional programs can ensure that such positions are occupied by leaders who possess a thorough grasp of the nature and potentialities of applied mathematics and are capable of making an efficient choice and use of facilities and personnel. In this connection it may be well to emphasize that in many cases it is not so much a question of organizing and blueprinting as it is one of creating a suitable climate.

While the pragmatic viewpoint is important, certainly the cultural aspects of mathematics should not be overlooked. The fundamental discoveries in pure mathematics must be looked upon as major contributions to our culture. We are all familiar with the famous paradox of Zeno illustrated by the story of Achilles and the tortoise. The object since antiquity of endless argumentation by philosophers, this paradox finds its clarification in the theory of infinite series. That it is essential for those in the applied fields to be acquainted with the more academic questions and the discipline of rigor of pure mathematics is not only true because this knowledge and experience may suddenly acquire unsuspected practical value, but it is also of importance simply for the same purpose that motivates the study of the liberal arts—the achievement of an intimate communion with the great historical currents of thought and the practice of a formative intellectual discipline.

Finally, if we wish to carry this viewpoint to its ultimate conclusion, it is to be hoped that the teaching of mathematics as a purely cultural subject, stripped of its technicalities and the narrow viewpoint of the specialist, may become a requirement in our general curriculum. The impact of the mathematical sciences on our society is becoming more evident every year, influencing the language and concepts of our culture, and expanding rapidly into fields that until today were looked upon as the exclusive domain of those trained in the traditional methods of economics, law, and business administration. If the means of accomplishment remain a question for debate, we should never lose sight of the fact that, in many ways, the final purpose is humanistic, and that intellectualism should never be promoted for its own sake or at the expense of the spiritual, the artistic, and the creative. Thus can we hope to meet the challenge of the future and fulfill its bold promise.

Some Recent Developments in Airfoil Theory

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SUMMARY

Several recent developments in airfoil and wing theory have as their goals the extension of classical methods to account for characteristically viscous phenomena. Airfoil theory has always recognized the existence of such phenomena as explanations of the presence of circulation and vortex wakes; these new investigations are attempts to include detailed descriptions in theoretical models or to extend classical models into areas of strong viscous effects, such as separation and stalling. Some of these studies follow directly from suggestions made by von Kármán, and others are reminiscent of his earlier research.

This review is concerned with investigations in four categories: (1) the theory of profiles with boundary layers in steady flow, (2) the theory of profiles with boundary layers in unsteady flow, including extensions of unsteady airfoil theory, (3) the theory of wings with leading-edge separation, and (4) Prandtl wing theory applied to partially stalled wings.

INTRODUCTION

THE THEORY of the flow of fluids past airfoils and wings has attracted the attention of fluid mechanicists for over half a century. Many of these workers have been mathematically inclined, and they have found the field a fertile one for application of their most elegant techniques. Von Kármán and Trefftz,[1] for example, generalized the famous Joukowski transformation in a most ingenious way to produce their families of airfoils having finite trailing-edge angles. The Prandtl lifting-line wing theory[2] and Munk's theory[3] of thin airfoils are also examples in this category, as are the extensions of these two subjects by von Kármán and his colleagues.[4, 5, 6] Von Kármán's treatment[7] of the lifting surface by Fourier-integral techniques is another ingenious and elegant mathematical accomplishment.

Nevertheless, airfoil theory, in spite of its considerable mathematical development, has always rested upon a distinctly physical base—namely, the production of circulation by viscous processes. The effects of these complicated processes are summed up, approximately, in the Kutta-Joukowski trailing-edge condition. In fact, the recognition of this essentially viscous origin of circulation, lift, and induced drag might be said to mark the emergence of aerodynamics as a science and to distinguish it from the purely mathematical fluid mechanics of the preceding century. Von Kármán has described these historic developments in a recent book.[8]

One of the first attempts to describe in more detail the viscous phenomena that produce circulation was made by von Kármán and Millikan,[9] who calculated the effect of Reynolds Number on the maximum lift coefficient of an airfoil. It was contemporary with the well-known work of Howarth,[10] who explained, by means of boundary-layer theory, the lift of a thin elliptic cylinder at incidence. These were investigations in which the viscous origins of lift were not only recognized but became the center of attention.

In the present paper, it is proposed to discuss only this viscous aspect of airfoil theory—i.e., its extension to problems involving strong viscous effects. This means to omit from the review all the modern achievements of the more classical, perfect-fluid theories of airfoils and wings in steady and unsteady motion in the several speed regimes. This sweeping omission may be the more acceptable, however, since these developments have recently been reviewed elsewhere by a number of distinguished authors.[11, 12, 13, 14]

THE THEORY OF PROFILES WITH BOUNDARY LAYERS: STEADY FLOW

The first class of investigations to be mentioned in this broad category stems directly from the work of Howarth, mentioned above. These constitute attempts to estimate the boundary-layer growth on airfoil profiles and to account for its effects on the steady-state value of the circulation and pressure distribution. Previously it had been suggested[15] that the profile be given a fictitious camber to make its circulation agree with the measured value, or even that the Kutta-Joukowski condition simply be discarded, the circulation put equal to its measured value, and the resulting infinite velocities simply be ignored.[16]

Preston[17] was the first successfully to employ detailed boundary-layer and wake calculations to predict the circulation of airfoils. A more recent investigation along the same line has been made by Spence,[18] who has achieved some simplification of Preston's technique. The calculation, like Howarth's, is one of successive approximations. Assuming first that the boundary-layer thickness is known, the potential flow outside the boundary layer is corrected for the displacement thickness of the boundary layer and the viscous wake by distributions of sources of proper strength along the airfoil contour and along the approximate position of the wake. Actually, of course, the boundary layer cannot be calculated until the external flow is known, and the circulation, which is an essential feature of the external flow, cannot be determined until the boundary layer is known and an adequate criterion—a refinement of the Kutta-Joukowski condition—is applied.

The criterion applied by Preston and Spence to determine the circulation is, essentially, that the pressure

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at the trailing edge (see Fig. 1) shall have the same value when determined from the potential-flow values above and below the airfoil. Preston pointed out that the pressure variations across the boundary layers are not negligible in such a singular region as that of the trailing edge. Thus, one does not equate the static pressures at points 1 and 2 in Fig. 1 at the edges of the upper- and lower-surface boundary layers, but instead one estimates the pressure increments, say Δp_1 and Δp_2, and requires that

$$p_1 - \Delta p_1 = p_2 - \Delta p_2 \tag{1}$$

where p_1 and p_2 are static pressures at points 1 and 2. But p_1 and p_2 are simply related to the velocities q_1 and q_2 at the same points, since these points are outside the viscous layers:

$$p_1 + (1/2)\rho q_1^2 = p_2 + (1/2)\rho q_2^2 \tag{2}$$

Thus, condition (1) becomes

$$\left(\frac{q_1}{q_2}\right)^2 = \frac{1 - \varpi_2}{1 - \varpi_1} \approx 1 + \Delta\varpi \tag{3}$$

where ϖ_1 denotes $\Delta p_1 \div (1/2)\rho q_1^2$, ϖ_2 denotes $\Delta p_2 \div (1/2)\rho q_2^2$, and $\Delta\varpi = \varpi_1 - \varpi_2$. Spence finds that $\Delta\varpi$ is small, of the order of 0.01. It is not, of course, inconsistent with boundary-layer theory to recognize the existence of pressure increments Δp in boundary layers on curved surfaces. In fact, this theory gives these increments but states that they are of such order as to be neglected in the momentum equation for the velocity component along the surface.

Howarth's[10] criterion to determine the circulation, on the other hand, was that the total flux of vorticity into the wake must be zero for steady flow—a theorem due to G. I. Taylor. By means of boundary-layer approximations, Howarth reduced this to the statement

$$q_1 = q_2 \tag{4}$$

which may not appear to be consistent with Eq. (3). There is no conflict, however, since Howarth's calculation needs only to be revised for the curved-surface boundary layer considered by Preston and Spence. Namely, the vorticity ζ in such a layer is given by[19]

$$\zeta = -(\partial u/\partial y) - \kappa u \tag{5}$$

where u is the component of velocity along the wall, y is the usual boundary-layer coordinate normal to the wall, and κ is the curvature of the wall. The vorticity flux, for a layer of thickness δ, is then

$$\int_0^\delta \zeta u \, dy = -\int_0^\delta \frac{\partial u}{\partial y} u dy - \kappa \int_0^\delta u^2 dy \tag{6}$$

The first term integrates immediately to give $-(1/2)q^2$, where q is the velocity at the edge of the layer, while the second, according to boundary-layer theory,[19] is just $-(1/\rho)\int_0^\delta (\partial p/\partial y)dy$. Thus,

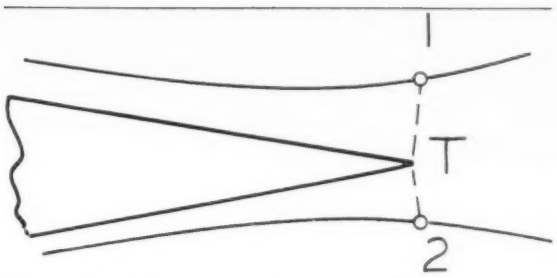

FIG. 1. Sketch showing boundary-layer configuration at trailing edge.

$$\int_0^\delta \zeta u \, dy = -(1/2)q^2 - \frac{1}{\rho}\Delta p \tag{7}$$

where Δp is, as before, the pressure rise across the boundary layer.

Consequently, Howarth's criterion of vanishing total vorticity flux becomes

$$(1/2)q_1^2 - (1/\rho)\Delta p_1 = (1/2)q_2^2 - (1/\rho)\Delta p_2 \tag{8}$$

which is exactly the same as Eq. (3).

It seems unnecessary to review Spence's calculation in detail here. In general outline, his method is to transform the profile conformally into a circle and to distribute sources along its surface to account for the displacement effects of boundary layer and wake, as mentioned above. Thus he obtains an expression equivalent to Eq. (3) relating the circulation to $\Delta\varpi$. Approximate boundary-layer methods are used to calculate, for any given potential flow, the trailing-edge thicknesses and the pressure increments Δp_1 and Δp_2. Spence makes specific suggestions regarding the practical procedure of iteration.

Spence's conclusion, based on comparison with experiment, is that his method is accurate when the boundary-layer thicknesses are known or can be accurately predicted—i.e., that the limitations of the theory are the limitations of boundary-layer knowledge—such as the prediction of transition points. This conclusion was arrived at by comparing the theoretical results with experimental values both for cases where measured boundary-layer data were used in the calculations and where available boundary-layer theories were used to predict these data.

Reference 18 also presents pressure distributions on a Joukowski airfoil at several Reynolds Numbers at 6° incidence. It is interesting that the distribution deviates only near the trailing edge from the inviscid-fluid distribution for the same value of the circulation. This is, of course, another justification of the proposals made by Pinkerton[15] and Weinig,[16] but the new theory goes much further in that it provides a method to predict this value of the circulation. Fig. 2, taken from reference 18, presents integrated results from these pressure distributions; it is seen that the viscous effect on hinge moments, particularly of a short-chord flap, is large.

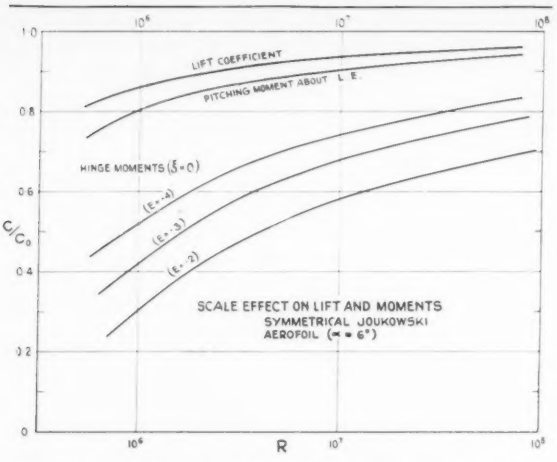

FIG. 2. Calculated scale effect on lift and moments. R denotes Reynolds Number based on chord, C the coefficient of lift, moment, and hinge moment, respectively, and C_0 the corresponding ideal-fluid value. E represents the ratio of flap chord to airfoil chord, and "$\xi = 0$" indicates that the flap is unbalanced (from reference 18).

Thus, two-dimensional steady airfoil theory has successfully been extended into the realm of fluids of small viscosity and it may be predicted that Spence's techniques will become tools for routine engineering investigations. Their extension to subsonic compressible flows seems to be straightforward.

Substantially the same procedures should handle problems where separation takes place forward of the trailing edge on the upper surface, as will certainly occur at higher angles of incidence. The complication introduced in this situation is that the separation point will be unknown as the calculation begins, and will have to be determined by successive trials, as in Howarth's problem. The accuracy of the method must decrease when the region of separated flow becomes large enough to affect the potential flow field appreciably, as Spence points out. Research on this interaction problem would seem highly desirable.

The Theory of Profiles with Boundary Layers: Unsteady Flow

Extension of these considerations to unsteady flow would have even greater practical value, since it would shed light on important unanswered questions. Such questions arise especially in the interpretation of classical unsteady airfoil theory and experiment and in assessing their validity at extreme values of frequency, Reynolds Number, and angle of attack. Unfortunately, this extension involves some formidable complications. To begin with, the criterion to determine the value of the instantaneous circulation requires some modification. The Taylor-Howarth criterion is surely not applicable without change, since the net rate of vorticity transport into the wake should not be zero but should correspond to the rate of change of the circulation about the profile. The Preston-Spence criterion regarding the trailing-edge pressure, on the other hand, must apply as well in unsteady as in steady flow, if separation does not occur forward of the trailing edge. Once again, these two criteria turn out to be identical.

Consider first the revised Taylor-Howarth criterion; for the case sketched in Fig. 1, one would require

$$\text{Rate of vorticity flux into wake} = d\Gamma/dt \tag{9}$$

where $\Gamma(t)$ is the circulation at time t. From the earlier calculation [Eq. (7)], then,

$$(1/2)\,(q_1^2 - q_2^2) - (1/\rho)\,(\Delta p_1 - \Delta p_2) = d\Gamma/dt \tag{10}$$

instead of Eq. (8).

The Preston-Spence criterion, for this case, would still be stated by Eq. (1), but, instead of Eq. (2), one has

$$\left(\frac{\partial \phi}{\partial t}\right)_1 + p_1 + \frac{1}{2}\,\rho q_1^2 = \left(\frac{\partial \phi}{\partial t}\right)_2 + p_2 + \frac{1}{2}\,\rho q_2^2 \tag{11}$$

where ϕ denotes the velocity potential of the exterior flow. Eq. (1) now becomes

$$\left.\begin{aligned} \frac{1}{2}\,(q_1^2 - q_2^2) - \frac{1}{\rho}\,(\Delta p_1 - \Delta p_2) &= \\ \left(\frac{\partial \phi}{\partial t}\right)_1 - \left(\frac{\partial \phi}{\partial t}\right)_2 &= \\ \frac{\partial}{\partial t}\,(\phi_1 - \phi_2) &= \\ \frac{d\Gamma}{dt} & \end{aligned}\right\} \tag{12}$$

as stated in Eq. (10).

Thus, if the instantaneous boundary-layer thickness and wake configuration were known, the process of calculating the potential flow and circulation should proceed by analogy with the steady flow case. The effect on the potential flow of the vortical wake would have to be calculated, as is done approximately in the classical theory of airfoils in unsteady motion.[20] However, the problem of estimating boundary-layer properties for given, unsteady, potential flow is considerably more difficult than its steady flow counterpart. This statement is correct for laminar layers; it is an understatement for the turbulent case. In the first place, one would not know where to put the transition point in unsteady flow; moreover, the customary practical procedures for estimating turbulent boundary-layer growth, based as they largely are on empirical relationships, can hardly be expected to apply without modification to unsteady flow.

If boundary-layer separation in unsteady flow is found to occur forward of the trailing edge, there are more complications in the calculation. Once again the rate of vorticity flux into the wake must be related to the rate of change of circulation [Eq. (9)]. But it is not easy to define the point (separation point), in unsteady flow, that divides the boundary layer from the wake. Certainly the usual criterion of vanishing shear at the body surface cannot be correct in general. Furthermore, the rate of motion of the separation point,

assuming it to be adequately defined, enters the vorticity criterion, for the pertinent flux is that across the dividing line:

$$\int_0^\delta \zeta \, u_{\text{rel.}} \, dy \tag{13}$$

where $u_{\text{rel.}}$ denotes the relative boundary-layer velocity component as seen by an observer moving with the separation point.

Elliptic Cylinder at Maximum Lift

There has been one serious attack upon this unsteady airfoil boundary-layer problem, made recently by Moore.[21] Inspired by the problem of rotating stall in axial compressors, he has extended Howarth's calculation for the elliptic cylinder with laminar boundary layer to the case where the cylinder is fixed but is blown upon by a stream of varying direction.

Moore has reduced the enormous complexity of the general problem in two ways. First, he considers only small rates of change of incidence $\dot{\alpha}$—periodic phenomena of low frequency, say—and therefore is entitled to carry terms proportional to $\dot{\alpha}$ while neglecting those proportional to $\ddot{\alpha}$ and all higher derivatives. Thus, his calculation is the first step in refinement of a purely quasi-steady theory. Second, he considers only the case of maximum (steady-state) lift; thus, the quasi-steady variations of circulation are zero.

The combined effects of these two limitations on the vortex flux and circulation criterion are profound: since there is no net quasi-steady effect on circulation, the largest effect must be proportional to $\dot{\alpha}$; thus $d\Gamma/dt$ can be at most of order $\ddot{\alpha}$ and can be neglected. Consequently, Moore's circulation criterion is again that the total vorticity flux must vanish. The complication represented by Eq. (13) remains, however, and his criterion becomes, instead of Howarth's [Eq. (4)],

$$\frac{1}{2} q_1^2 - q_1 \frac{ds_1}{dt} = \frac{1}{2} q_2^2 - q_2 \frac{ds_2}{dt} \tag{14}$$

where q_1 and q_2 are, as before, the potential-flow velocities at upper- and lower-surface separation points and ds_1/dt, ds_2/dt denote the rates of travel of the respective separation points.

The difficulty of defining the separation point in unsteady flow has been mentioned above. Moore uses the same criterion as in steady flow—i.e., vanishing of the wall shear—but advances it "without complete confidence." To be sure, this is the definition usually used in unsteady flow;[22] nevertheless, Moore's doubts seem to be legitimate. In fact, the present author suggests that the correct definition must be different; namely, the appearance of a stagnation point and a dividing flow trajectory between boundary-layer and wake fluids *in the flow seen by an observer moving with the separation point*. The reasoning behind this suggestion is simply that "separation" is presumed to mean the breaking away of the boundary layer from the surface, invalidating the boundary-layer approximations, and that (by analogy with the steady case) this will be caused by wake fluid thrusting itself under the boundary layer and deflecting it away. Thus the appearance of a dividing streamline (trajectory) between wake fluid and boundary-layer fluid should be a necessary feature of separation, and this can surely have meaning only when viewed from the moving frame of reference. Certainly the local, instantaneous vanishing of wall shear has no such significance.

If this proposal for a redefinition of unsteady separation is correct, Moore s results cannot be precise, although it is easily possible, as he suggests, that they may be correct to the order ($\dot{\alpha}$) that concerns him.

To finish the description of Moore's work, the following brief outline of his procedure is offered:

(a) Howarth's flow pattern and boundary layer for the elliptic cylinder at maximum lift ($\alpha = 7°$) constitute the basic flow, which is perturbed by variations of the direction of the oncoming stream.

(b) The perturbation of the potential flow is determined by classical methods. A perturbation of the circulation about the cylinder is assumed to occur.

(c) The accompanying small perturbations of the boundary layer are of two types—quasi-steady and unsteady. Both of these are calculated from momentum-integral formulas. The appropriate formulas for unsteady perturbations were provided by Moore in an earlier note.[23]

(d) These calculations relate the positions and velocities of the separation points to the circulation perturbation. The vorticity condition (14) is then applied, and this perturbation is determined.

The principal result of this investigation is the lift-hysteresis that occurs when the cylinder is exposed to an oscillatory airflow at its maximum-lift attitude. Moore's particular result, for the unsteady increment of lift coefficient is

$$-36 \, \dot{\alpha} l / U \tag{15}$$

where l is the chord (major axis) of the cylinder and U the stream speed, so that $\dot{\alpha} l / U$ is the dimensionless rate of change of incidence.

Moore seems to be surprised by the sign of this result, which, he says, is contrary to some experimental observations. To be sure, his result denotes counterclockwise hysteresis in the curve of lift versus angle of attack, whereas clockwise hysteresis at the stall is a familiar phenomenon in wind-tunnel experiments. To the present reviewer, this discrepancy is not surprising, since the commonly observed hysteresis arises from delays in separation and reattachment of the boundary layer which must be difficult to reproduce in a small-perturbation theory. In fact, the dominant effect in this hysteresis may arise from quasi-steady terms that Moore eliminated by considering small perturbations from maximum lift. His result is, no doubt, correct for sufficiently small oscillations about maximum lift, but these may be so small as to miss the phenomenon of clockwise hysteresis. In other words, the linearization achieved by the small-perturbation

hypothesis may be very restrictive when applied right at the point of maximum lift.

The result expressed in Eq. (15) obviously can be interpreted as a lagging of the lift behind the angle of attack; in fact, at the point of maximum lift treated, it represents exactly 90° phase lag behind a sinusoidal angle of attack. Phase lag independent of frequency but dependent upon mean incidence was first suggested as an empirical description of oscillating airfoil results in real fluids by Mendelson,[24] and has been adopted as an explanation of rotating stall by the present writer.[25] As has already been suggested above, investigations like Moore's seem highly desirable in view of the importance of the viscous phenomena in rotating stall, stall flutter, and many other problems, and in view of the difficulty of performing meaningful experiments in this field before the governing phenomena have been predicted theoretically.

Extensions of Unsteady Airfoil Theory

Prior to Moore's more ambitious investigation, it had occurred to other aerodynamicists[26] to carry over to airfoils in unsteady motion the simple ideas of "reduced circulation"—i.e., the relaxation of the Kutta-Joukowski condition that had been found useful in steady flow theory. It was suggested, for example, that the degree of departure from the Kutta-Joukowski value—the "efficiency" of circulation production—might be carried over from steady flow observations. Such an assumption, of course, would amount to by-passing the whole procedure of boundary-layer estimation and circulation determination which has been described above and carried out, under rather special assumptions, by Moore. The difficult question, however, is just how to carry over the steady flow data. Supposing the circulation is known as a function of angle of attack in steady flow, just what is the equivalent or appropriate angle of attack in unsteady flow?

An answer to this question has been proposed by Woods,[27] who takes the slope of the circulation curve to be a function of only the instantaneous, local angle of attack at the trailing edge. The justification of this assumption is hard to understand; it is presumably based on ideal-fluid considerations rather than on any argument concerning the boundary layer.

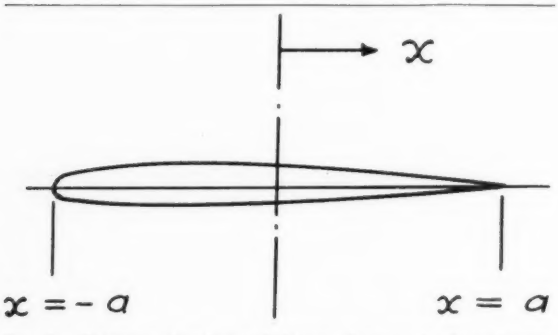

FIG. 3. Diagram showing definitions of terms in Rott-George theory.

Another proposal has been advanced by Rott and George.[28] They point out that, in general, a thin airfoil with arbitrary circulation, in either steady or unsteady motion, has a pressure coefficient distribution of the form

$$C_p(x) = \frac{S_l}{\sqrt{x+a}} + \frac{S_t}{\sqrt{x-a}} + f(x) \qquad (16)$$

where x is measured along the chord line from the midpoint (Fig. 3), the chord length is $2a$, $f(x)$ is regular at both leading and trailing edges, and S_l and S_t are constants determined by the motion, camber, and incidence of the airfoil and its circulation. Thus, in general, there are singularities at both leading and trailing edges, whose strengths are given by S_l and S_t respectively. The Kutta-Joukowski hypothesis is $S_t = 0$, while the value of S_l is not restricted; the leading-edge singularity is only a simplified representation of a strong, but finite, peak of the pressure distribution near a rounded leading edge.

To relax the Kutta-Joukowski condition means to permit S_t to assume a nonvanishing value. It is Rott's and George's basic assumption that the appropriate value of S_t in a real fluid depends on the value of S_l. The basis of this assumption is the following: The magnitude of S_t, being a measure of the deviation from the Kutta-Joukowski circulation, is actually a measure of the boundary-layer thickness at the trailing edge. The crudest approximation to this thickness would make it proportional to the pressure rise between leading and trailing edges; hence, to S_l. In unsteady flow, however, there must be a time lag between the value S_l and the value S_t that it determines—i.e.,

$$S_t(t+\tau) = F[S_l(t)] \qquad (17)$$

As their first attempt to apply this idea, Rott and George adopt the linear relationship

$$S_t(t+\tau) = -MS_l(t) \qquad (18)$$

where M is constant, since this is the familiar case of a linear relation between lift coefficient and incidence in steady flow. They carry over this relation to unsteady flow problems, such as oscillations of airfoils, determining M from the slope of the lift curve in steady-state tests at the mean incidence of the oscillation. They assume reasonable values for the time lag τ between 0 and $2a/U$.

George[29] has worked out a complete theory of unsteady two-dimensional airfoils on this basis. The results are at least qualitatively encouraging, since they show trends quite similar to those of experimental results, in comparison with classical thin airfoil theory.

One can suggest a possible generalization of the Rott-George idea to airfoils with hinged flaps. For such an airfoil, the form analogous to Eq. (16) is

$$C_p(x) = \frac{S_l}{\sqrt{x+a}} + \frac{S_t}{\sqrt{x-a}} + S_h \ln(x - x_h) + f(x) \qquad (19)$$

where the logarithmic term represents the singularity at the hinge line ($x = x_h$). A fitting generalization of Eq. (17) would then be

$$S_t = F(S_l, S_h) \tag{20}$$

(where the symbols denoting appropriate time lags are omitted for simplicity). In this case, two empirical relationships would be carried over from steady airfoil tests, such as the slopes of the curves of lift coefficient against angle of attack and flap angle, to determine the function F in Eq. (20).

It will be interesting to see whether approximations of the Woods or Rott-George type will be adequate to explain and predict the phenomena of stall flutter, rotating stall, etc. To ascertain this, it might be possible to carry out more elaborate calculations, like Moore's, and to interpret their results approximately in terms of these simpler assumptions.

The Theory of Wings with Leading-Edge Separation

It has been observed that wings with highly swept-back, sharp leading edges produce a kind of flow pattern, at moderate and large angles of attack, that differs considerably from the ones assumed in perfect-fluid, small-perturbation treatments according to classical wing theory. Specifically, the flow does not negotiate the sharp edge, but instead forms a vortex sheet there, which rolls up and is carried downstream, somewhat as sketched in Fig. 4. This is a typical case of "component flow separation" in the sense of three-dimensional boundary-layer theory;[30] it is not actual flow separation because the component of flow along the leading edge is not disturbed, and it is not, ordinarily, accompanied by large drag, buffeting, or other common manifestations of separation. In fact, as will be mentioned below, the phenomenon causes an increase, rather than decrease, of lift.

This is another case where strong viscous effects require that classical wing theory be modified. This was done semiempirically, using the cross-flow concept of the Munk-Jones wing theory, by Flax and Lawrence.[31] First to notice that an explicit theoretical result might be obtained for certain supersonic wings were Legendre[32] in France and Adams and Brown[33] at Langley Field. The theory was improved essentially by Brown and Michael[34] and Edwards.[35] It has been applied by Cheng[36] to the side edges of rectangular wings.

The cases that permit explicit solution are those that have the conical character. By the usual argument, in these cases all velocity components are constant along rays from the vertex; thus the strength of the vortex sheet shed at the leading edge and the rate of vorticity production are constant and the entire flow pattern, including the spiral vortex sheets, is similar at all chordwise stations.

The basic idea of Legendre, Adams, Brown, et al. is the familiar and powerful one of by-passing the details of vorticity production and treating the fluid as inviscid, with the vortex spiral present. They all had the further idea of approximating to the rolled-up sheet as a single vortex filament whose strength varies linearly downstream from the vertex. Then, by analogy with the classical theory of airfoils in unsteady flow, they replaced the complicated processes of vortex production by applying the Kutta-Joukowski condition at the leading edge.

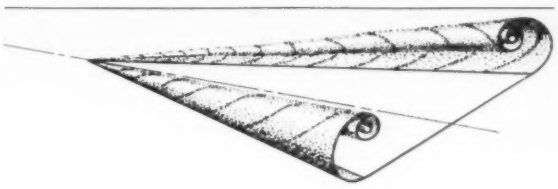

Fig. 4. Schematic drawing of a slender delta wing with leading-edge separation (from reference 34 by permission of NACA).

In principle, these assumptions are sufficient to determine the strength and position of the vortex filament and thus to permit a complete solution of the wing problem, for these are three unknowns (strength of the vortex and two coordinates of its position) and one has three conditions—the Kutta-Joukowski condition and the statement that the vortex must drift in a manner consistent with the flow velocity (two components). It was in the application of this last condition that complication ensued and the various investigators followed somewhat different paths.

The reason for the differences of opinion is basically that the replacement of the vortex sheet by a single vortex of nonconstant strength violates the laws of vortex conservation. The vortex must be fed vorticity from the leading edge and must therefore be connected to it by an "umbilical cord." In the single-vortex model, the details of this "cord," which appears simply as a barrier (in the mathematical sense) drawn between the leading edge and the vortex, are not important, but there is, nevertheless, a force acting upon it. Legendre and Adams-Brown neglected this force and assumed that the vortex filament itself was force-free and must therefore drift with the local flow velocity. As a consequence, they are confronted by the force on the barrier in their results, and this amounts to an appreciable ambiguity. For example, different values of lift are obtained by integration of surface pressures on the wing and by use of momentum principles, since one includes the barrier force and one does not.

The improvement due to Brown-Michael and Edwards was to recognize that the correct analog to the force-free character of the vortex sheet in the real flow is to require vanishing total force on *vortex and barrier* together, in the single-vortex model.

The results of these investigations are very encouraging. Edwards[35] finds, for example, the following formula for the lift coefficient of a flat delta wing at angle of attack α:

$$C_L = (1/2)A\alpha + \pi A^{1/3}\alpha^{5/3} \tag{21}$$

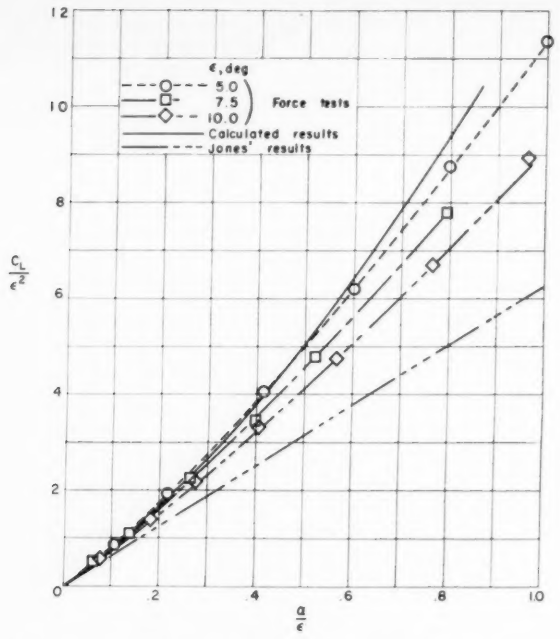

FIG. 5. Comparison of measured and calculated lift results for delta wings. ϵ denotes the vertex half-angle. The solid curve is the theoretical result of Brown and Michael[34] discussed here; "Jones' results" denotes the usual slender-wing approximation. Tests were made with 1° wedge section wings at a Mach Number of 1.9 (from reference 34 by permission of NACA).

where A is the aspect ratio, assumed to be small. The first term is the familiar result of Jones' slender wing theory, while the second is the correction for leading-edge vortex separation. Clearly, it represents an increase of lift compared to the classical case. Flax and Lawrence[31] and others have suggested a quadratic term for this correction; the Edwards result suggests that this may actually be an approximation to the 5/3-power term, for narrow conical wings. The results of Brown and Michael[34] are the same and they have also provided the subsequent term, which is proportional to $\alpha^{7/3}$. Adams[33] also found that the increment in lift due to vortex separation varies as $\alpha^{5/3}$, but there is an ambiguity in the constant of proportionality, as mentioned above. Legendre's results[32] for lift coefficient are wrong because of a mathematical error noticed by Adams.[33] In addition to the lift curve, Brown and Michael calculated some pressure distributions showing the effects of leading-edge vortices. These, too, exhibit at least qualitative agreement with experimental results.

Unfortunately, careful comparisons between experiment and theory disclose that quantitative agreement actually occurs only for very narrow wings, with vertex half-angles of 5°, for example.[34] A comparison of lift-coefficient variation with angle of attack is shown in Fig. 5. Further research by Michael[37] has cast light on the probable reasons for the discrepancy. His experiments show that for wider delta wings the leading-edge separation results in vortical flow regions over the upper surface but not the rolled-up vortex sheets. Even for the narrow wings, the rolled-up vortices seem to assume positions not in complete agreement with theoretical results. In some cases there appear to be numerous vortices, instead of the two assumed in the theory.

Rott[38] has studied the matter of vortex-sheet separation from the edges of plates and wedges and has come to the conclusion that the Kutta-Joukowski condition may be too rough an approximation, especially when the separation occurs at the edge of a wedge, rather than a plate. He also believes that a similar difficulty occurs when there is a general flow tending to carry the vortex sheet back over the edge from which it separates. This is what occurs in the case of the delta wing. Thus, by analogy, one may speculate that a more complicated leading-edge condition might be required for delta wings.

Thus, this new idea in wing theory cannot be said to be completely successful yet. The basic concept of recognizing vortex-sheet production at leading edges and treating it in the perfect-fluid approximation seems to be an important one, and one that may be useful in explaining a number of important phenomena. Its limitations, both as to geometrical shapes and Reynolds Numbers, will have to be determined by systematic experimental research.

PRANDTL WING THEORY APPLIED TO PARTIALLY STALLED WINGS

In this final section, it is proposed to report on another extension of wing theory into a realm of strong viscous effects, namely into the region of stalling. The investigation upon which these remarks are based is not actually a new one; it was carried out 10 years ago but has never been reported in published form. It stems directly from a suggestion made by Professor von Kármán at the California Institute of Technology.

Von Kármán noticed, some years ago, that the familiar integro-differential equation of Prandtl lifting-line wing theory,

$$\Gamma(y) = \frac{1}{2} Uc(y)m(y) \left\{\alpha(y) - \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\Gamma'(\eta)d\eta}{y - \eta}\right\} \quad (22)$$

has nonunique solutions when the lift-curve slope $m(y)$ is negative. (Here $\Gamma(y)$ denotes the spanwise distribution of circulation, U the speed of flight, $c(y)$ the chord, and $\alpha(y)$ the geometrical absolute angle of attack. The wing tips are at $y = \pm b/2$ and the integral is the Cauchy principal value.) In fact, if $\alpha(y) = 0$ and $m < 0$, there are infinitely many "eigen"-distributions of circulation $\Gamma(y)$, both symmetrical and antisymmetrical, for any plan form $c(y)$. The meaning of these is clear: they are the distributions whose accompanying downwash distributions exactly provide the induced angles to support them. Obviously, for $\alpha(y) \not\equiv 0$ these distributions can be added to a particular distribution, so that the solution is again ambiguous. For $m(y) > 0$ there are no such solutions. These facts were pointed out (some years after von Kármán's

discovery of them) in reference 39, where the "eigen"-distributions were actually calculated for certain classes of wings and became the basis for a quick method of solving Prandtl's equation for $m > 0$ and arbitrary $a(y)$.

This discovery suggested to von Kármán that the phenomenon of nonunique solutions might also occur for lift-versus-incidence curves having portions of positive and negative slopes, or having discontinuities. If this were true, the appearance of asymmetrical distributions and large rolling moments near the stall might be explained without postulating an initial rolling velocity. The usual textbook explanation, it will be recalled, is based on an initial rolling velocity that stalls one side of the wing and unstalls the other. It fails to explain a number of well-known facts about asymmetrical stalling, for example, the violently asymmetrical distributions and tremendous rolling moments observed by every wind-tunnel operator who tests a symmetrical wing firmly fixed to a set of balances and brings it to the stall.

As will be shown presently, von Kármán's surmise was correct. Before proceeding further, however, one must ask whether the Prandtl lifting-line theory, and Eq. (22) in particular, are valid for stalled and partly stalled wings. It seems clear that its accuracy will be substantially poorer than at small angles of attack. The small-perturbation hypothesis is not so good, and trailing vortices are likely to be appreciably displaced from the stream direction. The assumption of two-dimensional flow at each station y may be a crude approximation of the facts, and, if so, the section-characteristic curve (lift coefficient versus angle of attack) may not be transferable with complete accuracy from two- to three-dimensional flow. Nevertheless, as has already been mentioned above, the concept of circulation and the Kutta-Joukowski formula have been shown to be extensible to flows with viscous wakes, and numerical accuracy is not one of the goals of this investigation. Generally speaking, the differences between the validity of Prandtl's equation at the stall and at low incidence are quantitative differences; nearly all the approximations of the theory become poorer at large incidence, but the qualitative relationships must remain the same.*

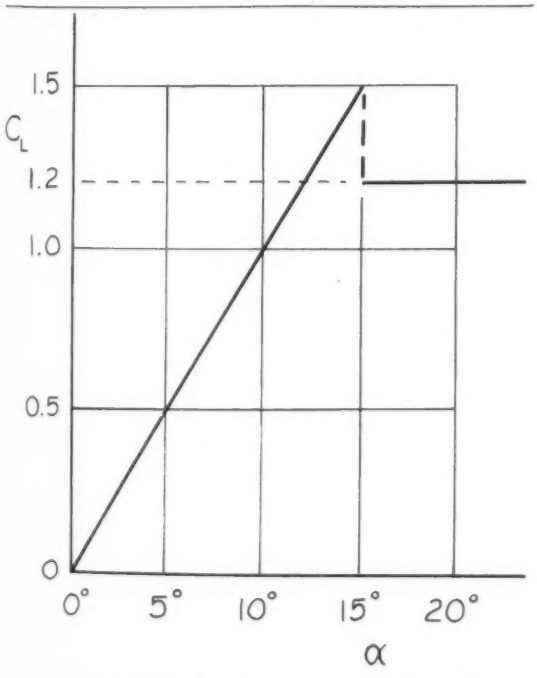

FIG. 6. Section lift-coefficient curve assumed for studies of partially stalled wings (reference 40).

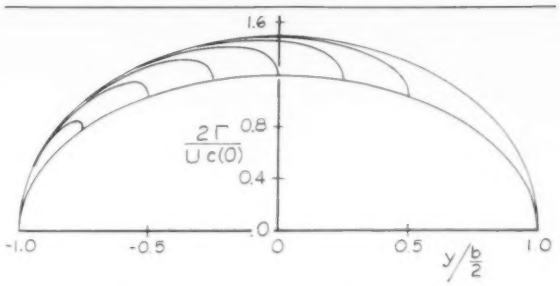

FIG. 7. Dimensionless circulation distributions for a flat elliptic wing of aspect ratio 10.19 having section characteristic shown in Fig. 6, according to Schairer.[40] The two symmetrical (elliptical) distributions are completely unstalled (upper) and stalled (lower); the asymmetrical distributions represent partial stalling at various intermediate angles of attack.

The investigation suggested by Professor von Kármán was carried out by Robert S. Schairer, under the writer's supervision, and is reported in his thesis.[40] The section characteristic assumed by Schairer is the discontinuous one shown in Fig. 6. He found it necessary to solve Eq. (22) by trial and error in this case, since the commonly used techniques failed. Most of the familiar methods of solution, of course, are based on the assumption that $m(y)$ does not vary with Γ. Among those that avoid this restriction is the procedure due to Tani,[41] which has been found particularly useful for wings flying at high subsonic speeds where $m(y)$ varies with the lift coefficient. It is clear that the present problem is simply an extreme one in this category; hence Tani's procedure should be useful.

As a matter of fact, Schairer's procedure was basically the same as Tani's, insofar as it consisted of assuming $\Gamma(y)$, calculating the induced-angle term [the integral in Eq. (22)], and thus the right-hand side of Eq. (22) for given $\alpha(y)$, and comparing the result with the assumed $\Gamma(y)$. The feature of Tani's procedure that failed was the successive-approximation procedure—i.e., Schairer found that the discrepancy between the assumed $\Gamma(y)$ and the calculated right-hand side often gave him no clue as to the modification to be made in the next trial. Thus the procedure of finding solutions was a tedious one.

Typical results of Schairer's analysis are shown in Fig. 7. These are lift distributions for an untwisted

* The Prandtl theory was applied to wings near and beyond the stall in an earlier investigation by Wieselsberger,[42] who considered only symmetrical distributions of circulation.

wing of elliptical planform, at several angles of attack. In these asymmetrical solutions, the wing has a constant lift coefficient (equal to 1.2—cf. Fig. 6) to the right of a certain station and a variable lift coefficient over the unstalled part to the left of that station. Schairer found these asymmetrical solutions within only a narrow range of angle of attack near the stall. Only one such solution (at most) was found for any given angle of attack. The choice between this solution and the classical symmetrical one must involve a question of relative stabilities of the two flows. No attempt has yet been made to solve this formidable stability problem.

Schairer's typical asymmetrical solutions produce large rolling-moment coefficients, reaching maximum values of the same order—e.g., 0.03—as are obtained by full aileron deflections. He made calculations for straight-tapered and twisted wings as well as the case of Fig. 7, but his results fail to show important quantitative differences between the behavior of these wings. He did not find larger rolling moments, for example, on a wing tapered 5:1 than on one tapered 3:1. In flight, as is well known, more highly tapered wings are more likely to exhibit tip stalling and violent wing dropping at the stall.

The present writer has verified that this phenomenon—the appearance of asymmetrical solutions to Prandtl's equation—occurs for a large variety of nonlinear lift-versus-incidence curves having discontinuities and or negative slopes, in addition to the particular one studied by Schairer. It is suggested that this characteristic of Prandtl's equation, especially in view of its clear physical meaning, may be of considerable theoretical significance. It has already suggested to two different investigators,[25, 42] by analogy, the possibility of asymmetric flow patterns occurring in another symmetrical geometrical situation, namely in an axial-flow compressor stage. There the equations to be satisfied are different, but the character of solutions and also their physical meaning are much the same. Once again the production of lift (pressure rise) by any part of the annular cascade results in a distribution of induced velocities all around the annulus. Asymmetric solutions appear when the induced-velocity distribution provides just the induced angles necessary to support the distribution of pressure rise. As in the wing problem, this occurs only when the blade-characteristic curve has a discontinuity or negative slope. There are various opinions on how to describe this blade characteristic, especially for cascades of high solidity; nevertheless, the basic idea described here is common to all the theoretical studies.[25] Marble,[42] for example, has used a discontinuous characteristic curve analogous to the curve of Fig. 6.

Conclusion

Several recent investigations have succeeded in extending the scope of wing theory beyond the restrictions of classical, small-perturbation, ideal-fluid approximations. Those discussed here are essentially of two categories: (a) attempts to describe in some detail the flow of a real fluid about a wing, including viscous phenomena, and (b) extensions of inviscid-fluid theory by consideration of some gross effect of viscosity, such as vortex separation and partial stalling. A number of pressing problems of practical aerodynamics seem to require such advances in wing theory. Progress to date along these lines is far from adequate; it is, nevertheless, encouraging and indicative of possible fruitful methods of attack to be used from here on.

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Inelastic Buckling—From a Designer's Viewpoint

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Introduction

It has been nearly 70 years since the first analysis was made of a stability problem in which the compressive stress was greater than the proportional limit of the material. The classical papers of Engesser, Considère, von Kármán, and Southwell,[1–6]† published between 1889 and 1912, extended the column theory of Euler into the inelastic or plastic range. By rationalizing the short column theory, these authors also strengthened the Euler presentation which was in some disrepute due to a lack of agreement with experiments in the short column range.

Although some papers continued to be published in this field after 1912, it was not until approximately 1940 that a new major attack was made on the problem. This was largely due to the increased efforts of the aircraft designer to achieve lighter and more efficient structures for the air frame. Still more recently a new parameter has been added to the problem, that of elevated temperature and the creep associated with high temperatures in structural materials. These temperatures arise from the aerodynamic heating that, in turn, is caused by the ever-increasing flight velocities of aircraft and missiles.

The problem of plasticity and plastic buckling is many-faceted, and the literature on it is extensive. Prager[7] has indicated that the field can be roughly divided into three major parts—namely, (1) the physical theory of plasticity, which is based on the general concepts of the structure of matter; (2) the mathematical theory of plasticity, which is concerned with a critical discussion of the fundamental stress-strain laws in the plastic range; and (3) the solution of concrete problems of stress and strain in the plastic range.

It is the third of the above fields of interest that will be discussed here. The questions that will be posed, and to which partial answers have been found, are the following: from an engineering design standpoint, what stability problems concerning stresses above the proportional limit have been solved to such an extent that design equations are available to the stress analyst; what are the orders of accuracy of these equations; and what areas need further experimental and/or theoretical investigation? The hypothetical engineer asking these questions is not, in general, concerned with the finer points involved in the physical or mathematical theories of plasticity but is primarily interested in finding analysis methods and techniques for determining stresses and deformations in the plastic range which will lead to conservative (but not excessively so) structural analyses.

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† The reference list at the end of this paper is necessarily very incomplete. Only those papers directly quoted in the text are shown.

Inelastic Buckling With No Creep

Structural materials, when subjected to stress above the proportional limit, will tend to have a continuously increasing deformation even though the load remains constant. This phenomenon, known as creep, normally takes place very slowly at room temperatures (except for certain stress ranges in the irons and low carbon steels) and, for loading times of a few tens or even hundreds of hours, can generally be neglected. As the temperature of the material is increased, not only does the modulus of elasticity and proportional limit decrease in value but the creep rate becomes accelerated until, at high temperatures, a material may fail under a constant, relatively low stress in a matter of a few minutes or even seconds. The term "high temperature" must be redefined for each material since what may be "high" for an aluminum alloy may be relatively "low" for a different material such as stainless steel. This section will only treat those cases where the creep rate is either so low, due to low temperatures and/or stresses, or where the loading time is so short that the maximum load is reached before creep has had time to appreciably affect the deformation of the structural element.

(a) *Stable Cross-Section Columns*

For an initially straight column, with a cross section which is not subject to local plate or shell buckling, axially loaded to a stress below the proportional limit, the value of the critical load is given by the Euler equation

$$P_c = c\pi^2 EI/L^2 = P_E \tag{1}$$

where c is a constant dependent upon the end restraints, E the modulus of elasticity, I the moment of inertia of the cross section, and L the length of the column. When the stress, given by $\sigma = P_c/A$ (where A is the cross-sectional area) is greater than the compressive proportional limit of the material, the Euler equation gives too high a value for P_c as compared with experiments in the short column range. One can, however, obtain satisfactory agreement with experiment if one

alters Eq. (1) to read

$$P_c = \eta\,(c\pi^2 EI/L^2) \tag{2}$$

where η is generally dependent upon the stress-strain curve of the material and the applied P_c/A stress.

Numerous theoretical or semiempirical models have been studied to obtain a value for η and a number of these are discussed in detail in reference 8. Although the various theories presented by such authors as Osgood,[9] Wang,[10] Shanley,[11] Drucker and Onat,[12] and others are of considerable scientific interest and are necessary for a complete understanding of the problem, it is generally found satisfactory for design purposes to set $\eta = E_T/E$ where E_T is the tangent modulus of the material at the P_c/A stress level. Thus the design equation becomes

$$P_c = c\pi^2 E_T I/L^2 = P_T \tag{3}$$

for all stress values since $E_T = E$ at stresses below the proportional limit. This equation will generally give slightly conservative values which will tend to compensate for small initial lacks of straightness or concentricity of loading of actual columns.

If any appreciable eccentricity of loading or lack of straightness is present, the P_c will only asymptotically approach the values given by Eq. (3) and the problem is more suitably treated as a beam column which is not an instability problem since the member is stable at all deflected positions under the proper loading. The behavior of an inelastic column with eccentric loading can be physically shown by considering the load deflection curve of an eccentric elastic column (See Fig. 1). In a long slender column, as the load increases [Fig. 1(a)] the deflection continues to increase until the sum of the direct compressive stress and the bending stress reaches the proportional limit of the material after which the load tends to decrease for an increase in deflection (dotted portion of the curves). For a shorter column with a higher value of P_E, the point where the combined stress reaches the proportional limit occurs at a smaller value of the deflection [Fig. 1(b)], and when the applied stress is in the inelastic range, there is a tendency for the load-deflection curve to take the form shown in Fig. 1(c). In both the elastic and inelastic columns the critical load of an initially straight column is given by the limiting case of an initially curved or eccentrically loaded column, however, the value of P_{max} for the elastic column asymptotically approaches P_c (at least until bending becomes important) while this is not true for the inelastic column. It is therefore seen that the effect of initial eccentricities is more important for the inelastic column problem. One method of calculating the load deflection curve for inelastic columns is given by Lin.[13] The effect of initial curvature is also treated in detail by Wilder, Brooks, and Mathauser.[14]

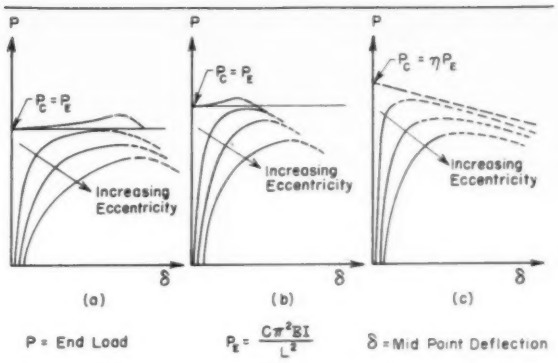

FIG. 1.

(b) *Plates Under Compression*

The elastic buckling equation of an initially flat plate subjected to a uniform compression along two edges is given by

$$\sigma_{cr} = [k\pi^2 E/12(1-\mu^2)](t/b)^2 \tag{4}$$

where k is a constant dependent upon the edge restraints of the plate, μ is Poisson's ratio, t the plate thickness, and b the plate width. To make this equation applicable in the inelastic range it is multiplied by a factor which is dependent upon the properties of the material throughout the stress range under consideration, or

$$\sigma_{cr} = \eta[k\pi^2 E/12(1-\mu^2)](t/b)^2 \tag{5}$$

Again, as was the case for inelastic columns, a number of attempts have been made to determine the proper value of η to use in Eq. (5). The most commonly known are those of Bijlaard,[15,16] Ilyushkin,[17] Gerard,[18] and Stowell,[19] all of which are essentially based on the deformation theory of plasticity. In addition one value based on a flow theory by Handelman and Prager[20] has been proposed. Fig. 2 taken from reference 21 shows the agreement between the various theories and a set of experiments carried out on thin-walled square tubes tested in compression in the inelastic range. It is interesting to note that the flow theory of reference 20 gives much poorer agreement with experiment than the deformation theories of the other authors. A possible explanation has been made on the basis of initial imperfections in the plate by Onat and Drucker in reference 22.

It is believed that Stowell's value for η—namely,

$$\eta = \frac{E_S}{E}\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{4} + \frac{3}{4}\frac{E_T}{E_S}}\right) \tag{6}$$

(where E_S is the secant modulus at σ_{cr}) will give reasonably good agreement with experiments for long, simply supported plates loaded beyond the proportional limit and can, therefore, be used for design purposes. Bijlaard's paper[16] gives more general equations and these have been put into chart form by Krivetsky.[23]

Preliminary work on effective widths of stiffened plates in the inelastic region as carried out at the National Luchtvaartlaboratorium in Amsterdam[24,25] indicate that the theoretical effective width in the elastic region can be used safely in the plastic region at least for ranges of $\sqrt{\epsilon_{cr}/\epsilon}$ between 0.28 and 0.75 for

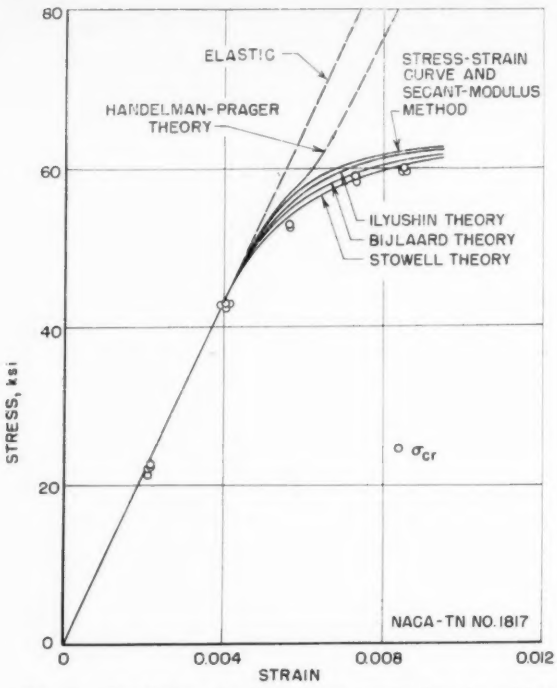

FIG. 2. Correlation of adjusted test results for the critical compressive stress with calculated plate buckling curves for long, simply supported flat plates of 14S-T6 aluminum alloy. (Calculated elastic critical compressive strain used for plate tests.)

materials comparable to 2024S aluminum alloys ($\epsilon_{cr} = \sigma_{cr}/E_s$, $\epsilon = \sigma/E_s$). More testing in other alloys and materials is needed so as to check the generality of the methods proposed.

(c) *Plates Under Shear*

As was the case for plates in compression, the elastic equation for the critical stress of a plate subjected to shearing stress can be modified to take into account stresses above the proportional limit. The resulting equation is

$$\tau_{cr} = \eta[k_s\pi^2 E/12(1 - \mu^2)](t/b)^2 \qquad (7)$$

where k_s is a coefficient based on the edge restraint and aspect ratio of the plate. Stowell[26] proposes a general expression for η as a function of the shear stress-strain curve of the material, the angle of the shear waves, and the edge restraint. Gerard proposes[27] that η be given by

$$\eta_s = G_s/G \qquad (8)$$

and justifies his assumption on tests on 24S-O sheet tested in shear (see Fig. 3). Other authors have proposed that

$$\eta_s = E_T/E \qquad (9)$$

which in general gives lower values than either Stowell or Gerard. Krivetsky[23] gives charts of Bijlaard's results[16] which may also be useful for designers. Since the experimental data is very limited, it is tentatively suggested that Eq. (9) be used until further experimental tests are available on different materials and different types of edge support which may lead to the justification of one or more of the other values of η. Use of Eq. (9) will generally give conservative results for all common structural materials.

(d) *Thin-Walled Columns*

Columns which are made up of plate or shell elements—which, by themselves, are subject to local buckling—pose a considerably more difficult problem. Stowell and Pride[28] have considered the problem of a column cross section made up of a web plate and n outstanding flanges ($n = 2$ for Z and channel sections, $n = 4$ for H sections) and, by using the plate buckling theory of Stowell,[19] are able to set up an analysis method for this type of structure which is in good agreement with experimental tests. Fig. 4 shows the agreement of their method with tests on an extruded H-section column of 75S-T aluminum alloy in which the buckling stress is carried nearly to the ultimate compressive stress of the material. The method proposed is one of trial and error but convergence is comparatively rapid if reasonable care is used in choosing the first approximations.

Another approach to the problem is given by Needham in reference 29. In this he establishes crippling values of plate columns by considering them as being built up of angles having two types of edge support. For example a channel section would consist of two angles, each having one edge free and one edge elastically restrained (the elastic restraint being the connection of the two angles in the center of the channel web), whereas a rectangular tube would be made up of four angles, each of which would have both edges elastically restrained. Curves of allowable loads for such angles are presented and good agreement is shown with experimental data. For intermediate column lengths,

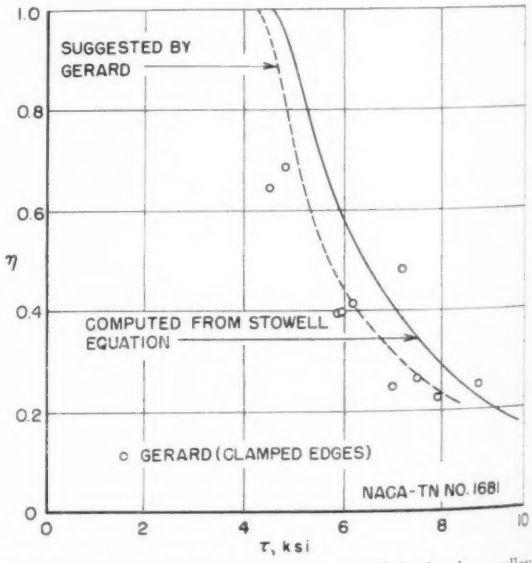

FIG. 3. Values of η for a long plate of 24S-O aluminum alloy under uniform shear stress.

the crippling stress, σ_{cc}, could be used in some intermediate column formula—such as the Johnson parabolic equation—to obtain column allowables up to the Euler curve.

(e) *Other Inelastic Buckling Problems*

Bijlaard and Krivetsky[16,23] also give equations for and plots of the plastic coefficient η for cylindrical shells under axial compressive loads and spherical shells under uniform external pressure. The value of the critical stress (or pressure) in the inelastic region is given by

$$\sigma_{cr_{inelastic}} = \eta \sigma_{cr_{elastic}} \tag{10}$$

in each case.

In reference 30 Wittrick treats the case of the lateral instability of a deep but thin rectangular beam when the stresses in the beam are above the proportional limit. This paper points out the effect of the shape of the stress-strain curve on the problem by comparing the lateral buckling of a material like an aluminum alloy which does not have a yield point with one like a low carbon steel in which a definite yield point is present.

Bijlaard[31] presents a discussion of the elastic and inelastic buckling of sandwich plates based on the theory of split rigidities. In this paper, plastic coefficients η are plotted for various cases of sandwich construction, and these will be found to be of great value to the designer considering the use of sandwich construction in the inelastic region.

A note by Klein[32] proposes a rapid means of determining the elastic-plastic buckling of tapered columns subjected to a distributed axial load along their length in addition to axial end loads. This corresponds to a stringer attached to a sheet structure which feeds axial loads into the stringer by means of a shear field in the sheet. The method proposed is semiempirical in nature but appears to give good agreement with limiting cases where exact solutions are known. It essentially uses a weighted $P/E_T I$ diagram for the tapered column utilizing only three cross sections of the column as controls. Accuracies of the order of a few per cent are indicated.

Inelastic Buckling With Creep

When columns and plates are loaded in compression and, at the same time, are subjected to elevated temperatures, one must consider the possibility of creep entering the problem. Therefore new parameters which need attention are: (1) the rate of loading (all discussion will assume that the rate of loading is slow enough so that dynamic impact stresses are negligible); (2) the time during which the member is to be subjected to the applied load; (3) the creep rates (compressive) of the material as a function of time, stress, and temperature; and (4) a decision as to the deformation magnitudes that will be considered as having caused failure of the structural element. An excellent review of the influence of creep in structural problems is presented by Hoff in reference 33 and should be recommended reading for all designers and stress analysts working with materials at elevated temperatures. In the sections of this paper which follow, only the major conclusions concerning specific problems will be discussed, again from the viewpoint of the stress analyst who needs design information for structural analysis purposes.

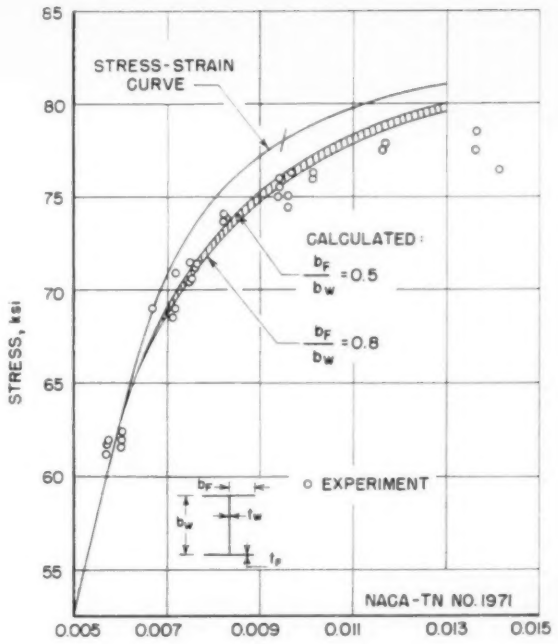

FIG. 4. Correlation of buckling curves and experimental results for extruded H-section columns of 75S-T aluminum alloy of constant thickness, which fail by local instability.

(a) *Stable Cross-Section Columns*

If a column is assumed to have initially a small eccentricity, upon loading the following deformation history occurs: Immediately upon application of the load, the deformation is increased from its initial value by an amount corresponding to the elastic deformation plus the deformation caused by the primary creep of the material. These deformations will, in general, be relatively small unless the actual applied load is nearly equal to the inelastic buckling load of the column. If the load is then held constant, the deformation will increase at a rate dependent upon the creep rate of the material at the given temperature and stress level. Since increased deformation alters the stress in the column, there is a dependency of creep rate and deformation to be considered. At some time after load application the column deflections will either increase until they are no longer tolerable in the struc-

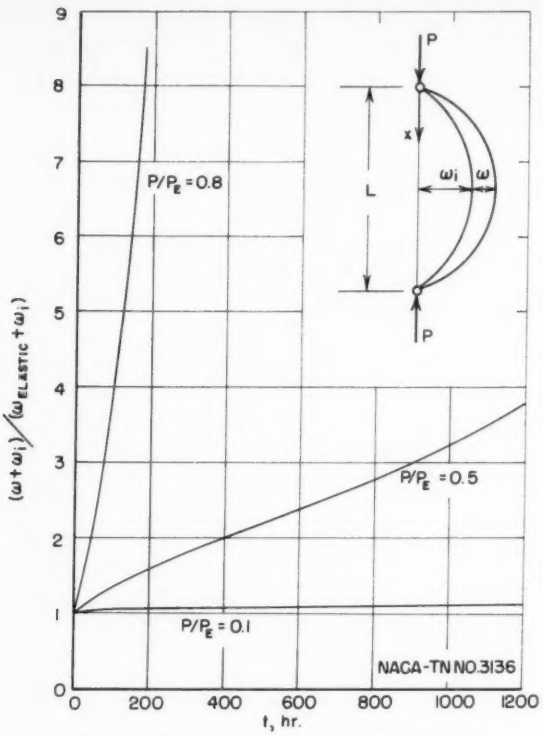

FIG. 5. Creep deflections of a simply supported initially curved column (linear viscoelasticity).

ture (deformation limiting) or they may rapidly approach infinity giving rise to what might be called a critical time. The time for the column to reach a critical or excessive value of deformation may be as long as a number of years (corresponding to concrete columns) or may only be a matter of a few seconds (which may be permissible for missile structural elements whose integrity is only required for very short times).

Kempner and Libove[34–37] indicate that where the material can be considered to be linearly viscoelastic the deformation of the column continues to increase at a reasonable rate with time and never tends to infinity at finite small times. See Fig. 5. On the other hand, if the material's viscoelasticity is nonlinear, there will be a finite time for each loading condition at which the column deflection will approach infinity. See Fig. 6. Experimental data of references 36, 38, and 39 would indicate that most aircraft structural materials will fall in the nonlinear class and that there will be a finite critical time at which very large deflections of a buckling type will occur and that curves of the form of Fig. 6 can be expected.

Higgins[38] presents a method of calculation of column creep curves which, while lengthy, is adaptable to machine calculation and which agrees quite well with experimental tests carried out on 75S-T aluminum-alloy columns tested at 600°F. The experimental results quoted in reference 39 show qualitative agreement with the curves obtained theoretically in reference 40 by Kempner and Patel. If additional experimentation proves the validity of this last reference, then the curves therein will offer a rapid means of determining the critical times for inelastic columns at high temperatures where creep is important.

In all of the above, it is seen that an accurate determination of the initial straightness of any column subjected to creep will be of great importance since the critical time to failure is vitally affected by the amount of initial eccentricity. For example, in the tests of Higgins[38] two columns carrying the same P/A stress of 2,500 psi at 600°F arrived at the same lateral deflection in times of 40 min. and 1,000 min., respectively. The one deflecting more slowly had an initial center deflection of 0.002 in. while the one that moved more rapidly had an initial deflection of 0.0068 in. Thus, a difference in initial deflection of less than five-thousandths of an inch in a column $^1/_4$ by $^3/_8$ in. in cross section gave a life difference of 25 times.

It must also be pointed out that even small changes in applied stress (all other variables being held constant) will give large changes in time to a critical deflection. Hoff[33] shows that for aluminum alloy at 600°F a change in average stress of 10 per cent (from 4,000 psi to 4,400 psi) may lower the critical time from 110 min. to 50 min. In view of the above two conditions, it is evident that either extreme accuracy or large margins of safety are required in the analysis of columns subjected to creep.

(b) *Cylinders*

In reference 41, Baer has discussed the plastic creep buckling of relatively thin-walled cylinders at high temperatures. The cylinders, which were analyzed and then checked experimentally, had an R/t ratio of 80, were made of magnesium alloy, and were tested at temperatures up to 600°F. The analysis is based upon six general assumptions—namely,

(1) That the tension and compression isochronous stress-strain curves are the same at all temperatures.

(2) The creep relationship is of the form

$$\epsilon = (\sigma/E) + A\sigma^m + B\,\sigma^n T + \epsilon''_{3max}\,(1 - e^{-\alpha T}) \quad (11)$$

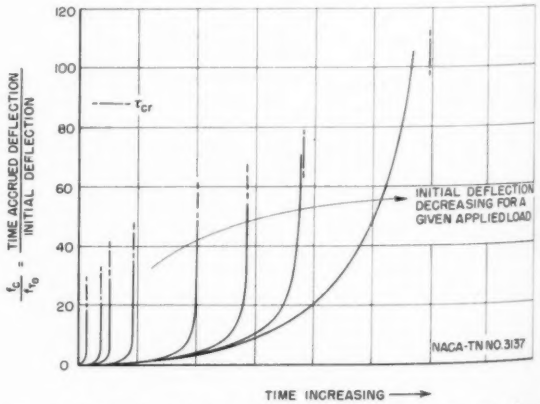

FIG. 6. Variations of deflections accrued for $t > 0$ with time parameter (nonlinear viscoelasticity).

where the unknown terms in the equation can be found from tension creep tests.

(3) The buckling follows the principle of inelastic action as formulated by Shanley.[11]

(4) The tangent modulus at a given stress is an explicit, even if unknown, function of the time at the given stress.

(5) If a buckling equation containing the tangent modulus exists, a time-dependent tangent modulus may be substituted to transform the buckling relation into a creep-buckling relation.

(6) For the cylinder, it is assumed that the buckling equation can be written as

$$\sigma_{cr} = 0.3E_T(t/R) \tag{12}$$

Carrying out the analysis and comparing the predicted buckling values with test data indicates that the predicted buckling values are conservative by relatively high percentages at the higher temperatures but that they agree very well at the lower temperatures. In general, however, the agreement is quite satisfactory for the times considered (1 to 60 sec.).

In addition, this report gives strain-time data on six magnesium, three aluminum, two titanium, and two steel alloys which may be of considerable value to the stress analyst.

Discussion and Conclusions

As a result of this survey of the problems of inelastic buckling, the writer is impressed by the great amount of work that has been done in the field since the work of the pioneers Considère, von Kármán, and Engesser who, it must be realized, actually came very close to the heart of the problem in their early papers. The bibliography attached to this article contains only a small fraction of the papers that have been written, and, if the author has missed any of major importance to the designer, he can only state that the slight was not intentional.

For the problem of inelastic buckling of columns without creep, the designer has available theoretical and experimental information which is sufficient to enable him to make a satisfactory analysis of structures containing such elements. For plates under compression and shear there is also available a considerable store of information, although, particularly for shear buckling, additional test data covering a wide range of materials would be of considerable value.

One area in which both theoretical and especially experimental data appears to be lacking is that of the thin-walled cylinder subjected to axial loads beyond the proportional limit. Bijlaard[16] has proposed an inelastic coefficient, η, for this case but no experimental data checking this coefficient has been found. This statement is not to be taken as offering doubt to any theoretical study of such a problem but actual test specimens and structures may deviate in important manners from the structures assumed for analysis purposes. The effects of these deviations must be known to the stress analyst and can, in general, only be found by testing.

The state of knowledge concerning inelastic buckling with creep is not nearly so comprehensive nor complete. This is to be expected since the problems of major structural elements operating at high temperatures is relatively new—at least in structures where the weight is as important as it is in aircraft and missiles. Basic concepts have been established for simple, stable cross-section columns, and one case of a thin-walled cylinder has been analyzed. Plates under compression, shear, and combined loads need attention as do columns made up of plate and shell elements since such structural elements comprise a large part of most aircraft and missile air frames. The extension of the cylinder problem to the case of a thin-walled cylinder subjected to axial load, internal pressure, bending, and torsion while, at the same time, being subjected to elevated temperature is becoming an even more pressing problem to the aircraft designer as airplanes and missiles approach higher and higher velocities with the resulting greater heating rates from aerodynamic causes. The next few years, therefore, should prove extremely fruitful for those research workers who are seeking difficult and, at the same time, useful problems which will utilize the highest possible skills in mathematics, applied physics, and the engineering sciences.

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A Review of Source Superposition and Conical Flow Methods in Supersonic Wing Theory

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Introduction

At the end of World War II the developments in aircraft power plants reached the point where it became feasible to construct vehicles—first rockets and guided missiles and, later, manned aircraft—which were capable of flight at supersonic speeds. At that time it became essential for the aerodynamicist to apply in design practice the seemingly much more complex theory of a compressible fluid medium rather than the classical theory of an incompressible fluid which had been sufficiently accurate for almost all problems only a decade earlier. The engineering application of the techniques of analysis which had been developed in the transition decade suddenly disclosed the unexpected result that many practical engineering problems that had been extremely difficult and impractical to solve in the incompressible fluid theory were relatively simple for supersonic speeds. Under these fortunate circumstances the aerodynamicist has been able to keep pace with the revolution of technology which has been required for the achievement of supersonic flight.

It is of interest to note that the comprehensive review of the aerodynamics of a compressible fluid by G. I. Taylor and J. W. Maccoll,[1] which was published in 1935, presents the one- and two-dimensional nonviscous theory of supersonic motion in a form which is essentially unchanged today. In particular they discussed Ackeret's[2] linearized two-dimensional theory, the two-dimensional application by Prandtl,[3] Ackeret,[4] and Busemann[5] of the method of characteristics for the integration of a hyperbolic partial differential equation by use of the hodograph transformation. They also treated the simplest flow with a degenerate hodograph, following Prandtl's[6] and Meyer's[7] solution of the expansion around a two-dimensional corner. The corresponding solution for the flow past a circular cone, which had been treated by Taylor and Maccoll,[8] was the only three-dimensional example considered.

Since supersonic aerodynamic configurations generally involve slender bodies and relatively low aspect ratio wings, three-dimensional effects are of primary importance in practical engineering problems. Several concepts which were introduced at the time that reference 1 was being written lead to the additional methods on which most of the current engineering theory is based. First, von Kármán and Moore[9] applied the linearized theory to axially symmetric bodies and solved the problem by the superposition of supersonic sources. Second, Busemann[10] introduced the concept of a conical flow, which is a three-dimensional generalization of the Taylor and Maccoll cone solution since the degenerate hodograph is a function of two variables instead of one. Finally, von Kármán, in a paper presented at the Fifth Volta Congress which had the problems of high-speed flight as its general theme, developed a general theory of the drag of slender axially symmetric bodies and considered the solutions for a family of bodies related by a simple affine transformation in Cartesian coordinates to obtain a similarity theory for the linearized equations of motion. The combinations of these various concepts and their logical extensions form the basis of most of the current aerodynamic design methods. It is the purpose of the present paper to review some of the phases of supersonic aerodynamic theory which have developed from these origins. Many of the problems discussed here have also been treated in his recent, much more extensive, review of linearized theory by G. N. Ward.[12] It is hoped that this simple treatment, written from a different viewpoint, will be a useful supplement.

Basic Relations

Consider the irrotational, inviscid, isentropic flow of a perfect gas which has a uniform stagnation enthalpy. Under these conditions a velocity potential Φ will exist such that the velocity vector u_i is given by†

$$u_i = \partial\Phi/\partial x_i \tag{1}$$

Suppose in addition that there is a region at infinity where the flow is a uniform flow of velocity U in the direction of the positive x axis. In this region let $\partial\Phi/\partial t = 0$. Under these conditions the exact equation satisfied by the velocity potential (neglecting body force field effects) is

$$\frac{\partial^2\Phi}{\partial t^2} + 2\frac{\partial\Phi}{\partial x_i}\frac{\partial^2\Phi}{\partial x_i \partial t} + \frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\frac{\partial^2\Phi}{\partial x_i \partial x_j} = a^2\frac{\partial^2\Phi}{\partial x_i \partial x_i} \tag{2}$$

where a is the local speed of sound. If c is the speed of

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† A Cartesian coordinate system (x, y, z) is assumed initially. In addition a Cartesian tensor subscript notation, with the summation convention, is used.

sound in the undisturbed flow,

$$a^2 = c^2 + \frac{1}{2}(\gamma - 1)\left(U^2 - \frac{\partial \Phi}{\partial x_i}\frac{\partial \Phi}{\partial x_i} - 2\frac{\partial \Phi}{\partial t}\right) \quad (3)$$

where γ is the ratio of the specific heats.

If the perturbation added to the uniform flow at infinity is represented by the perturbation potential φ, then

$$\Phi = Ux + \varphi \quad (4)$$

If only first-order terms in the perturbation potential are retained, the differential equation which it must satisfy is thus

$$\frac{\partial^2 \varphi}{\partial t^2} + 2U\frac{\partial^2 \varphi}{\partial x \partial t} + U^2\frac{\partial^2 \varphi}{\partial x^2} = c^2\frac{\partial^2 \varphi}{\partial x_i \partial x_i} \quad (5)$$

This is the linearized equation for nonstationary perturbations. If the perturbation is not time-dependent, this reduces to the simpler Prandtl-Glauert equation,

$$(M^2 - 1)\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (6)$$

where $M = U/c$ is the Mach Number in the undisturbed flow.

In order to develop a suitable set of elementary solutions of the Prandtl-Glauert equation, it is convenient to introduce a transformed set of coordinates which are appropriate if $M > 1$. Let $\beta = \sqrt{M^2 - 1}$ and

$$\left.\begin{aligned} r &= [x^2 - \beta^2(y^2 + z^2)]^{1/2} \\ \mu &= \{1 - \beta^2[(y^2 + z^2)/x^2]\}^{-1/2} \\ \theta &= \tan^{-1}(y/z) \end{aligned}\right\} \quad (7)$$

In these coordinates the Prandtl-Glauert equation becomes

$$r^2\frac{\partial^2 \varphi}{\partial r^2} + 2r\frac{\partial \varphi}{\partial r} + \frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial \varphi}{\partial \mu}\right] + \frac{1}{1 - \mu^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0 \quad (8)$$

It may be noted that this equation is identical to the usual representation of the Laplace equation in polar coordinates. The harmonic solutions, obtained by the separation of variables, are thus

$$\varphi = \sum_{m,\,n} A_{m,\,n} \begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} \begin{Bmatrix} P_n^m(\mu) \\ Q_n^m(\mu) \end{Bmatrix} \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \quad (9)$$

For any fixed value of the separation parameter n, this solution is a homogeneous function of order n in the Cartesian variables. If

$$\varphi = r^n f_n(\mu, \theta) \quad (10)$$

the differential equation satisfied by f_n is

$$(1 - \mu^2)\frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial f_n}{\partial \mu}\right] + \frac{\partial^2 f_n}{\partial \theta^2} + n(n + 1)(1 - \mu^2)f_n = 0 \quad (11)$$

The most elementary solution of Eq. (9) (with $m = n = 0$) is

$$\varphi_1 = 1/r \quad (12)$$

Since the coordinate r is imaginary except in the fore and aft Mach cones, the solution may be considered to be zero except in the aft Mach cone ($x > 0$). It thus represents a disturbance originating at the origin. This is the simple supersonic source solution. The solution in the fore Mach cone, if it were used, would represent a disturbance originating at infinity which is cancelled by the singularity at the origin.

A set of solutions for the nonstationary case can be developed in a similar manner. If the space coordinates are changed to (r, μ, θ) by Eq. (7) and the time variable is transformed to

$$\tau = \beta^2 ct - Mx \quad (13)$$

Then Eq. (5) becomes

$$r^2\frac{\partial^2 \varphi}{\partial \tau^2} = r^2\frac{\partial^2 \varphi}{\partial r^2} + 2r\frac{\partial \varphi}{\partial r} + \frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial \varphi}{\partial \mu}\right] + \frac{1}{1 - \mu^2}\frac{\partial^2 \varphi}{\partial \theta^2} \quad (14)$$

This is identical to the classical wave equation in spherical coordinates. The solutions obtained by separation of variables are

$$\varphi = \sum_{l,\,m,\,n} A_{lmn} \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \begin{Bmatrix} P_n^m(\mu) \\ Q_n^m(\mu) \end{Bmatrix} \times \begin{Bmatrix} r^{-1/2}J_{-n-(1/2)}(lr) \\ r^{-1/2}J_{n+(1/2)}(lr) \end{Bmatrix} e^{\pm ll\tau} \quad (15)$$

The solution which corresponds to the supersonic source is obtained by $m = n = 0$, $l = \nu/\beta^2 c$, $A_{lmn} = \sqrt{\pi l/2}$, i.e.,

$$\varphi_2 = \sqrt{\frac{\pi l}{2r}}\, J_{-1/2}\left(\frac{\nu r}{\beta^2 c}\right) e^{i\nu\tau/\beta^2 c} \quad (16)$$

or, if the Bessel function is replaced by its trigonometrical equivalent,

$$\varphi_2 = \frac{1}{r}\cos\left(\frac{\nu r}{\beta^2 c}\right) e^{i\nu[t - (Mx/\beta^2 c)]} \quad (17)$$

Again, if this solution is to represent a source, it must be taken to be zero except in the aft Mach cone.

It may be seen that both Eqs. (12) and (17) were obtained with $n = 0$ (or $n = -1$), and that the corresponding solution of Eq. (11) which was used was $f_0 = 1$. It is apparent that both of these solutions can be generalized by multiplying by any function $f_0(\mu, \theta)$ which satisfies Eq. (11). For $n = 0$, Eq. (11) is

$$(1 - \mu^2)\frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial f_0}{\partial \mu}\right] + \frac{\partial^2 f_0}{\partial \theta^2} = 0 \quad (18)$$

If the conical variable μ is transformed to

$$s = \sqrt{(\mu - 1)/(\mu + 1)} \quad (19)$$

or

$$s = \frac{\beta\sqrt{y^2 + z^2}}{x + \sqrt{x^2 - \beta^2(y^2 + z^2)}} \quad (20)$$

Eq. (18) becomes

$$s\frac{\partial}{\partial s}\left(s\frac{\partial f_0}{\partial s}\right)+\frac{\partial^2 f_0}{\partial\theta^2}=0 \tag{21}$$

This is the standard form of the two-dimensional Laplace equation in polar coordinates. The general solution can thus be written most simply in terms of the complex variable

$$\zeta = se^{i\theta} \tag{22}$$

or

$$\zeta=\frac{\beta(z+iy)}{x+\sqrt{x^2-\beta^2(y^2+z^2)}} \tag{23}$$

The general solution of Eq. (21) is thus the real part of an arbitrary analytic function of the complex variable ζ—i.e.,

$$f_0 = Rl\{F_0(\zeta)\} \tag{24}$$

The simplest solutions are given by the various powers of ζ. Thus a family of solutions is given by

$$f_0=\sum_m B_m\begin{Bmatrix}s^m\\ s^{-m}\end{Bmatrix}\begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{25}$$

where, by Eq. (20),

$$\left.\begin{aligned}s^m&=\left[\frac{x-\sqrt{x^2-\beta^2(y^2+z^2)}}{\beta\sqrt{y^2+z^2}}\right]^m\\ s^{-m}&=\left[\frac{x+\sqrt{x^2-\beta^2(y^2+z^2)}}{\beta\sqrt{y^2+z^2}}\right]^m\end{aligned}\right\} \tag{26}$$

It is apparent that Eq. (26) is merely the explicit representation of the Legendre functions of Eqs. (9) and (15) for $n = 0$. If the functions of Eq. (25) are used to generalize the source solution of Eq. (12), the basic set of solutions is obtained which were used by Ward[13] to extend von Kármán's slender body theory to bodies of arbitrary cross section. The general solution indicated by Eq. (24) is essentially that which was first given by Donkin in 1857 for the general solution of the Laplace equation which is homogeneous and of degree zero in the Cartesian coordinates. This solution is the basis of the conical flow theory. It will also be shown that this solution can be used to construct solutions of Eq. (11) for any positive integral value of n. The conical flow theory provides the example for $n = 1$.

Source Superposition Methods

Following von Kármán's suggestion in his Volta Congress paper, Puckett[14] used a planar distribution of sources to solve several of the fundamental problems of three-dimensional wings. If sources [see Eq. (12)] of strength $A(\xi, \eta)$ are distributed over the plane $z = 0$, then the velocity potential at a point (x, y, z), where $z > 0$, is

$$\varphi=\iint_S\frac{A(\xi,\eta)d\xi d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2z^2}} \tag{27}$$

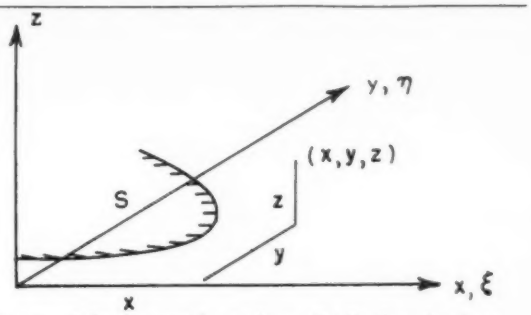

FIG. 1. Influence zone for a planar distribution of sources.

where the area S (see Fig. 1) over which the integration must be performed is bounded by the intersection of the forward Mach cone from the given point and the plane $z = 0$—i.e.,

$$0=(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2z^2 \tag{28}$$

If these limits are introduced into Eq. (27), it becomes

$$\varphi=\int_{x_1}^{x-\beta z}d\xi\int_{y-(\delta/\beta)}^{y+(\delta/\beta)}\times\frac{A(\xi,\eta)d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2z^2}} \tag{29}$$

where x_1 is the (finite) upstream limit of the disturbances and

$$\delta=\sqrt{(x-\xi)^2-\beta^2z^2} \tag{30}$$

In order to determine the boundary condition satisfied by this velocity potential, it is convenient to transform the variable in the inner integral. Let

$$\beta(y-\eta) = -\delta\sin\theta \tag{31}$$

Then Eq. (29) becomes

$$\varphi=\frac{1}{\beta}\int_{x_1}^{x-\beta z}d\xi\int_{-\pi/2}^{\pi/2}A\left(\xi,y+\frac{S}{\beta}\sin\theta\right)d\theta \tag{32}$$

Since $\partial\delta/\partial z = -\beta^2 z/\delta$

$$\frac{\partial\varphi}{\partial z}=-\pi A(x-\beta z,y)-\beta z\int_{x_1}^{x-\beta z}\frac{d\xi}{\delta}\int_{-\pi/2}^{\pi/2}\times\frac{\partial}{\partial\delta}\left[A\left(\xi,y+\frac{\delta}{\beta}\sin\theta\right)\right]d\theta \tag{33}$$

From this, if $A(x, y)$ is bounded,

$$\lim_{z\to+0}(\partial\varphi/\partial z) = -\pi A(x,y) \tag{34}$$

Thus any planar problem which specifies the normal velocity over a region R of the plane $z = 0$ determines the velocity potential at every point for which the region S [see Eq. (28)] is in R.

Puckett used this solution to check the two-dimensional theory and to calculate the symmetrical conical distributions which correspond to triangular wedges, and he applied these results to compute the lift and drag characteristics of several families of delta wings. If the procedure is applied to more complex shapes, the

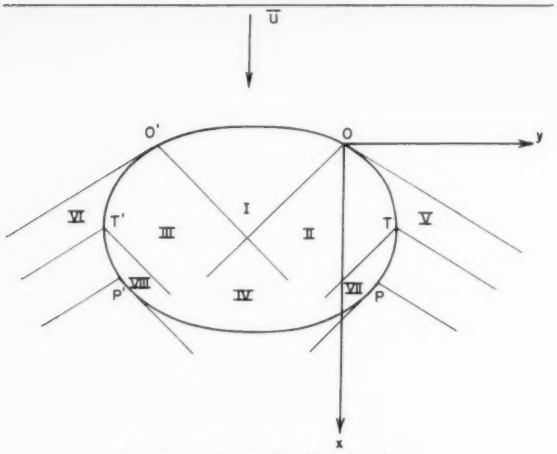

FIG. 2. Supersonic wing plan form.

integration procedures become very complex, even though it is only necessary to determine the velocity potential in the plane $z = +0$, and other procedures are usually employed.

This method breaks down if any of the edges of the wing are subsonic. The procedure for using the source superposition method in this case was developed by Evvard.[15] Consider a general wing plan form as shown in Fig. 2 where the diagonal lines are the Mach lines through the various points of tangency and the points T and T' are the wing tips. Then OO' is a supersonic leading edge, OT and $O'T'$ are subsonic leading edges, TP and $T'P'$ are subsonic trailing edges, and PP' is a supersonic trailing edge. The procedure used by Puckett will determine the solution in Region I. In Regions V and VI there will exist upwash fields which are initially unknown so the method cannot be directly applied to Regions II, III, or IV. In addition trailing vortex sheets will be shed at the trailing edges, and these sheets must be accounted for in determining the flow in Regions VII and VIII. Evvard's original solution applied to Regions II, III, and IV. The modifications necessary for Regions VII and VIII were developed by Ward.[16] The analysis for a point in Region II only will be presented here.

Consider a point Q in Region II (see Fig. 3). The characteristic net is also introduced as a set of generally nonorthogonal coordinates. As a matter of convenience, it will be supposed that the wing has zero thickness.* The boundary conditions which must then be applied are

$$\left.\begin{aligned} \partial\varphi/\partial z &= -U\alpha(x, y) \text{ in Region II} \\ \varphi &= 0 \qquad\qquad \text{in Region V} \end{aligned}\right\} \tag{35}$$

where $\alpha(x,y)$ is the local angle of attack. The second condition must be modified for a point in Region VII. By Eqs. (27) and (34), the velocity potential at the point Q is

$$\varphi = \frac{U}{\pi}\iint_{S_1+S_2} \frac{\alpha(\xi, \eta)d\xi d\eta}{\sqrt{(x_Q - \xi)^2 - \beta^2(y_Q - \eta)^2}} \tag{36}$$

where S_1 is the portion of the wing in the forward Mach cone and S_2 is the portion of Region V in the forward Mach cone. The flow downwash angle $\alpha(x,y)$ is unknown in Region V and must be determined from the second boundary condition. Let D be a point in Region V on the Mach line through Q. Then

$$0 = \iint_{S_1'+S_2'} \frac{\alpha(\xi, \eta)d\xi d\eta}{\sqrt{(x_D - \xi)^2 - \beta^2(y_D - \eta)^2}} \tag{37}$$

where S_1' and S_2' are the corresponding areas for the point D. Eq. (37) is thus an integral equation and must determine $\alpha(x,y)$ for Region V so that this condition can hold for any point D.

To solve this problem, introduce the characteristic coordinates (u, v) where

$$\left.\begin{aligned} u &= (M/2\beta)(x - \beta y) \\ v &= (M/2\beta)(x + \beta y) \end{aligned}\right\} \tag{38}$$

In characteristic coordinates Eq. (37) becomes

$$0 = \frac{1}{M}\int_0^{u_D} \frac{du}{\sqrt{u_D - u}}\left\{\int_{v_1(u)}^{v_D} \frac{\alpha(u, v)dv}{\sqrt{v_D - v}}\right\} \tag{39}$$

where $v = v_1(u)$ is the equation of the leading edge OO'. The outer integral is of Abel's type; consequently the inner integral must vanish identically—i.e.,

$$\int_{v_1(u)}^{v_D} \frac{\alpha(u, v)dv}{\sqrt{v_D - v}} = 0 \tag{40}$$

If the equation of the wing tip is $v = v_2(u)$, Eq. (40) may be separated into two parts,

$$-\int_{v_1(u)}^{v_2(u)} \frac{\alpha(u, v)dv}{\sqrt{v_D - v}} = \int_{v_2(u)}^{v_D} \frac{\alpha(u, v)dv}{\sqrt{v_D - v}} \tag{41}$$

where the left-hand side is a known function from the boundary condition for Region II. Eq. (41) is then an Abel-type integral equation for the determination of $\alpha(u,v)$ in Region V. The result can readily be written down; however, it is not necessary to do so in this case. Eq. (40) supplies the required result.

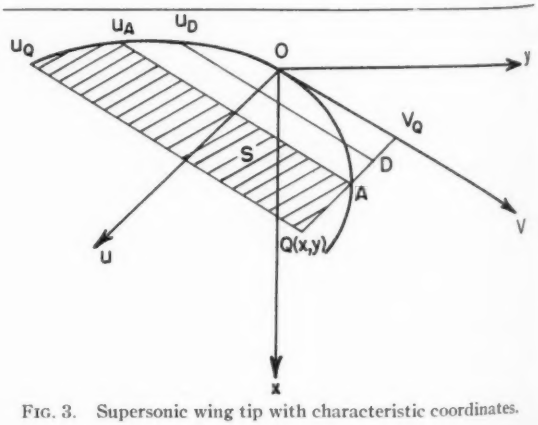

FIG. 3. Supersonic wing tip with characteristic coordinates.

* The effects of lift and thickness can be separated because of the linear approximation. For the thickness problem the normal velocity in Region V is zero because of symmetry.

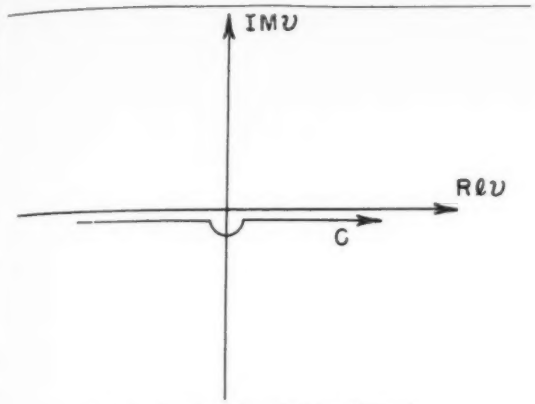

FIG. 4. Contour for unit-step integral.

If Eq. (36) is also written in characteristic coordinates, it becomes

$$\varphi = \frac{U}{\pi M}\int_0^{u_Q}\frac{du}{\sqrt{u_Q - u}}\left\{\int_{v_1(u)}^{v_Q}\frac{\alpha(u, v)dv}{\sqrt{v_Q - v}}\right\} \quad (42)$$

Since $v_Q = v_D$, Eq. (40) shows that the inner integral vanishes for all values of u less than u_A where the point A is on the wing leading edge—i.e., $v_2(u_A) = v_Q$. If the lower limit of the outer integral is thus increased to u_A, Eq. (42) is then seen to be the usual source superposition integral; however, only the sources in the shaded region S of Fig. 3 are considered. The upwash in S_2 of Region V thus compensates exactly for the remaining sources in S_1 of Region II. This result is generally referred to as the "Equivalent Area Theorem."

Source superposition methods have also been used to determine the solution to nonstationary flow problems. The analysis follows parallel lines except that the oscillating source [Eq. (17)] must be used in place of the steady-state source [Eq. (12)]. The velocity potential, corresponding to Eq. (27), for a distribution of strength $A(\xi,\eta)$ is then

$$\varphi = e^{i\nu t}\iint_S A(\xi, \eta)\exp\left\{-\frac{i\nu M}{\beta^2 c}(x - \xi)\right\}\times \cos\left(\frac{\nu r}{\beta^2 c}\right)\frac{d\xi d\eta}{r} \quad (43)$$

where, as before,

$$r = \sqrt{(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2} \quad (44)$$

and the region of integration S is given again by Fig. 1 and Eq. (28). The same procedure can then be followed to show that the boundary condition satisfied on the plane $z = +0$ is

$$\lim_{z\to+0}(\partial\varphi/\partial z) = -\pi A(x, y)e^{i\nu t} \quad (45)$$

No special restrictions on the nature of the separation parameter ν have yet been placed although it has generally been applied for real values of the parameter. Since it could be complex, a Fourier superposition can be used to obtain a solution suitable for "unit-step" boundary conditions. The "unit-step" function $H(t)$ where

$$H(t) = \begin{pmatrix}1 \text{ if } t > 0\\ 0 \text{ if } t < 0\end{pmatrix} \quad (46)$$

can be represented by the complex integral

$$H(t) = \frac{1}{2\pi i}\int_C e^{i\nu t}\frac{d\nu}{\nu} \quad (47)$$

where the contour C is the real axis, indented below at the origin, as shown in Fig. 4. A "unit-step" source can thus be constructed from Eq. (17) as follows:

$$\varphi_3 = \frac{1}{2\pi i r}\int_C \cos\left(\frac{\nu r}{\beta^2 c}\right)e^{i\nu[t-(Mx/\beta^2 c)]}\frac{d\nu}{\nu} \quad (48)$$

Since the cosine term in Eq. (48) can be written in exponential form, Eq. (48) can be evaluated in terms of the function $H(t)$. Thus

$$\varphi_3 = \frac{1}{2r}\left\{H\left(t - \frac{Mx}{\beta^2 c} + \frac{r}{\beta^2 c}\right) + H\left(t - \frac{Mx}{\beta^2 c} - \frac{r}{\beta^2 c}\right)\right\} \quad (49)$$

The factor which multiplies $1/r$ in Eq. (49) has a very simple geometrical interpretation. Draw a sphere of radius ct with its center a distance $Mct = Ut$ downstream from the source point. This sphere is tangent to the Mach cone at $x = \beta^2 ct/M$ (see Fig. 5). Then the multiplier has the value 1 ahead of the sphere (Region I), the value $^1/_2$ in the sphere (Region II), and the value zero behind the sphere (Region III). If the same superposition of Eq. (47) is performed on Eq. (45), it is thus seen that a superposition of these "unit-step" sources of strength $A(x, y)$ in the plane $z = 0$ satisfies the boundary condition

$$\lim_{z\to+0}(\partial\varphi/\partial z) = -\pi A(x, y)H(t) \quad (50)$$

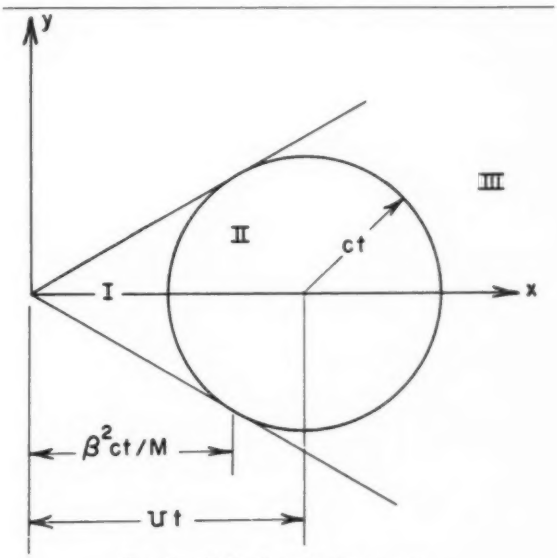

FIG. 5. Field of a unit-step source.

These methods yield especially simple solutions for two-dimensional problems. For example, suppose a two-dimensional wing has its leading edge at $x = 0$ and that the boundary condition on the wing is

$$\lim_{z \to +0} \frac{\partial \varphi}{\partial z} = w_0(x)e^{i\nu t} \tag{51}$$

Then by Eqs. (43) and (45) the velocity potential at a point $(x, 0, +0)$ is

$$\varphi = -\frac{e^{i\nu t}}{\pi} \int_0^x w_0(\xi) \exp\left\{-\frac{i\nu M}{\beta^2 c}(x - \xi)\right\} \times \int_{-(x-\xi)/\beta}^{(x-\xi)/\beta} \cos\left(\frac{\nu}{\beta^2 c}\sqrt{(x - \xi)^2 - \beta^2\eta^2}\right) \times \frac{d\eta}{\sqrt{(x - \xi)^2 - \beta^2\eta^2}} \tag{52}$$

The inner integral is readily seen to be a Bessel function, and

$$\varphi = -\frac{1}{\beta} e^{i\nu t} \int_0^x w_0(\xi) J_0\left[\frac{\nu}{\beta^2 c}(x - \xi)\right] \times \exp\left[-\frac{i\nu M}{\beta^2 c}(x - \xi)\right] d\xi \tag{53}$$

This is the general formula given by Temple and Jahn,[17] Miles,[18] and others.

This solution obviously applies not only to the two-dimensional case, but also to a family of wings which have straight leading edges and no subsonic edges. The result can also be generalized to determine the lift of those wings, such as the high aspect ratio delta wing, which have a supersonic leading edge and a straight trailing edge. In order to develop this result, it is necessary to consider a simple plunging motion of a two-dimensional wing for which w_0 is a constant. For this case, Eq. (53) becomes

$$\varphi = -w_0 \frac{\beta c}{\nu} e^{i\nu t} \int_0^{\nu x/\beta^2 c} J_0(u)e^{-iMu}\, du \tag{54}$$

According to the linearized theory, the pressure at any point on the wing is given by

$$p = p_0 - \rho[(\partial\varphi/\partial t) + U(\partial\varphi/\partial x)] \tag{55}$$

Define a new function $g(x)$ in terms of the local lift per unit of area as follows:

$$-2(p - p_0) = w_0 g(x)e^{i\nu t} \tag{56}$$

The function $g(x)$ then gives the spatial lift distribution for a unit w_0. This function can be readily computed from Eqs. (54) and (55); however, its explicit representation is not required for the present purpose.

Next consider a two-dimensional wing, Wing A, which has a chord d and the vertical velocity distribution

$$w_0(x) = \begin{cases} 0 & \text{if } 0 < x < a \\ w_0 & \text{if } a < x < a + da \\ 0 & \text{if } a + da < x < d \end{cases}$$

where w_0 and a are constants $(a < d)$ and da is a first-order infinitesimal. The zero downwash region at the front of the wing obviously has no effect on the flow problem and can be ignored. On the other hand, the aft portion does have an influence since it can support lift induced by the disturbance at $x = a$. It is obvious that the flow pattern of Wing A can be produced by superposition by subtracting the solutions of two wings, each having a uniform w_0, with the first having a chord d-a and the second having a chord d-a-da. It thus follows from Eq. (56) that the total lift per unit span, a first-order infinitesimal, of Wing A is

$$dL = w_0 g(d - a)e^{i\nu t} da \tag{57}$$

The function $g(d\text{-}a)$ thus can be considered to be the influence function giving the total lift for a unit disturbance at the point a. If a two-dimensional wing has an arbitrary distribution $w_0(\xi)$, it is possible to represent this by an infinite sum of wings like Wing A, and the total lift is

$$L = e^{i\nu t} \int_0^d w_0(\xi)g(d - \xi)d\xi \tag{58}$$

i.e., the influence function for the lift at station ξ is the same as the local lift per unit area at station ξ if the wing were in a reversed flow with a flat plate plunging motion duplicating that at station ξ.

Now Wing A can also be constructed by superimposing two three-dimensional wings, Wing B and Wing C, where Wing B has the w_0 distribution of Wing A for $y > 0$ and has $w_0 = 0$ for $y < 0$. The distribution for Wing C is the reflection in the plane $y = 0$ of the distribution for Wing B. The two wings obviously have the same lift characteristics; so, for each one, the total lift per unit span of the portion with $w_0 \neq 0$, even in the region near $y = 0$ where the flow is three-dimensional, is still given by Eq. (57).

Furthermore, if the solution for Wing B is modified by shifting the coordinate system parallel to the y axis and if this resulting flow pattern is subtracted from that of Wing B, it is seen that the lift per unit area, dL/da, is still properly expressed by Eq. (57), even if there are spanwise variations in w_0. There have been no restrictions placed on the distribution of w_0, so any finite wing which can be converted to an infinite span wing of constant chord without changing the flow pattern by adding plates with $w_0 = 0$ can be treated in this manner. Thus for any wing having a straight trailing edge and no subsonic edges (as, for example, the wide delta wing of Fig. 6), the total lift is given by

$$L = e^{i\nu t} \iint_S w_0(\xi, \eta)g(d - \xi)d\xi d\eta \tag{59}$$

where the area S is the total wing plan form. It must be remembered that while Eq. (59) gives the total lift, it gives no information regarding the lift distribution. If the wing is a flat plate so that w_0 is a constant, Eqs. (56) and (59) show the interesting result that the lift of the wing is the same for a reversed flow as it is for the direct flow. These results, summarized by Eq. (59), are special examples of the "reverse flow" theorems

which have been developed by von Kármán,[19] Flax,[20] and others.

The problem of applying oscillating source superpositions to lifting surfaces with subsonic edges, following Evvard's treatment of the steady state problem, has been investigated by Li and Stewart.[21] In order to simplify the treatment, the analysis will be given, in part, for a point in Region II of Fig. 2 for a wing of zero thickness. In this case the boundary conditions, corresponding to Eq. (35), are

$$\left.\begin{aligned} \partial\varphi/\partial z &= -U\alpha(x, y)e^{i\nu t} && \text{in Region II} \\ \varphi &= 0 && \text{in Region V} \end{aligned}\right\} \quad (60)$$

The velocity potential at any point in the plane $z = +0$ is given [see Eqs. (43) and (45)] by

$$\varphi = -\frac{U}{\pi} e^{i\nu t} \iint_{S_1+S_2} \alpha(\xi, \eta) \times \exp\left[-\frac{i\nu M}{\beta^2 c}(x-\xi)\right] \cos\left(\frac{\nu r}{\beta^2 c}\right) \frac{d\xi d\eta}{r} \quad (61)$$

where S_1 and S_2 are, the same as in the steady case, the areas in the forward Mach cone on the wing and in Region V, respectively, and r is given by Eq. (44). The function $\alpha(x,y)$ is known on the wing but is not known in Region V. On the other hand for a point D in Region V (see Fig. 3), the second boundary condition can be applied, and, corresponding to Eq. (37),

$$0 = \iint_{S_1'+S_2'} \alpha(\xi, \eta) e^{[-(i\nu M/\beta^2 c)](x_D-\xi)} \cos\left(\frac{\nu r_D}{\beta^2 c}\right) \frac{d\xi d\eta}{r_D} \quad (62)$$

This expression is, again, an integral equation for determining $\alpha(\xi, \eta)$ in Region V.

If characteristic coordinates [see Eq. (38)] are introduced, Eq. (62) can be written

$$0 = \int_0^{u_D} \frac{e^{[-(i\nu/\beta c)](u_D-u)}}{\sqrt{u_D-u}}\, du \left\{ \int_{v_1(u)}^{v_D} \alpha(u, v) \times \frac{e^{[-(i\nu/\beta c)](v_D-v)}}{\sqrt{v_D-v}} \cos\left[\frac{2\nu}{\beta c M}\sqrt{(u_D-u)(v_D-v)}\right] dv \right\} \quad (63)$$

From Eq. (63), it is apparent that, if the cosine term in the inner integral is replaced by unity, the equation can only be satisfied if the inner integral is identically zero for all values of u in the range of integration. The vanishing of the inner integral for any point D in Region V is the condition that justifies Evvard's "equivalent area" rule in the steady-state case. The same conclusion can be drawn here; namely if Evvard's "equivalent area" rule is applied to the oscillating case, the resulting velocity potential is correct for terms of order 0 and 1 in ν. Explicit formulas have been worked out for the corrections to the higher order terms;* however, the analysis is tedious and will not be duplicated here. The analysis was applied to the problem of the rectangular wing, which had been first solved by a different method by Miles.[22] The complete theory, including the correction terms, gave results in agreement with those obtained by Miles; however, it was noted that the correction terms were of a very small magnitude. Since most supersonic problems involve relatively low reduced frequencies, it should be necessary to compute only a small number of the correction terms. For very low frequencies, such as occur in dynamic stability problems, the "equivalent area" rule can be applied directly.

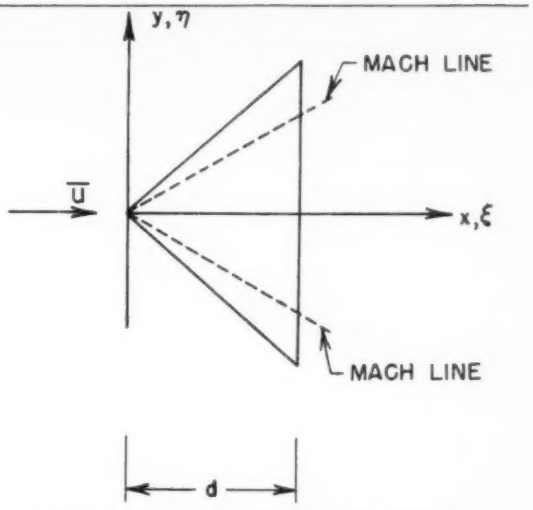

FIG. 6. Delta wing with supersonic leading edges.

CONICAL FLOW METHODS

Busemann observed that if the forward portion of a body is of a conical shape, then a change of scale in Cartesian coordinates transforms that part of the body into itself. On the other hand, the expressions for the boundary conditions, expressed in terms of the velocity components, is unchanged by the transformation if the flow is supersonic. This requires [see Eq. (10)] that the velocity potential be a homogeneous function of order 1 in Cartesian coordinates. He introduced the term "conical flow" to designate flows having this degree of homogeneity. In his original paper, Busemann[10] studied the conical solutions of the exact equation. Later, he applied the same concepts to the linearized equations of motion[23] and obtained the first correct solution of the steady-state flow near the tip of a rectangular flat plate wing. The method of analysis used here is essentially the modification of Busemann's analysis which was presented by Stewart.[24]

If the linearized velocity potential is a homogeneous function of order 1, it must be expressible [see Eqs. (10) and (11)] in the form

$$\varphi = rf_1(\mu, \theta) \quad (64)$$

While a simple family of solutions of Eq. (11) for $n = 1$

* Through an error in analysis, Li and Stewart in earlier papers had concluded that the result was also correct for the higher order terms (see reference 21 for details).

can be obtained by separation of variables—as in Eq. (9)—these solutions can be superimposed only with great difficulty to solve most of the practical problems. On the other hand, the Cartesian velocity components must be homogeneous functions of order zero which satisfy a simpler equation, Eq. (21). Since this is the Laplace equation in two dimensions, the well-known methods of conformal transformation can be used to determine the appropriate solutions. It thus appears desirable in the study of conical flows to approach the problem directly in terms of the velocity components rather than in terms of the velocity potential.

If some portion of the conical geometry being considered projects outside the Mach cone from the origin —e.g., the supersonic leading edges of a wide delta wing—it is then necessary [see Eq. (20)] to consider solutions in a region where the radial coordinate s of Eq. (21) is complex.

It is possible to carry out this analysis directly (see, for example, reference 12). Eq. (21) can be reduced to the one-dimensional wave equation in the outer region. In many important problems, the boundary conditions in the outer region correspond to flat plate wings. In this case the outer region is a swept two-dimensional flow, and the solution is most simply obtained by the sweep-back concept introduced by Busemann[25] in his Volta Congress paper. The solution in the outer region can then be used to determine the conditions at the Mach cone through the origin and thus to determine the boundary conditions for the inner region where s is real and conformal transformation methods can be used. For this reason, only the inner region where the solution is governed by an elliptic type differential equation will be considered here.

Since each of the three Cartesian components of the velocity vector in a conical flow must be homogeneous functions of order 0 and must satisfy the Prandtl-Glauert equation, each can be written as the real part of an analytic function of the complex variable. Thus,

$$u = Rl\{U(\zeta)\} \quad v = Rl\{V(\zeta)\} \quad w = Rl\{W(\zeta)\}$$

The three analytic functions are not independent, for the velocity components must satisfy the vorticity relations,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{65}$$

and the continuity equation,

$$\beta^2 \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{66}$$

Since u, v, and w are functions of only the two conical variables μ and θ, Eqs. (65) and (66) can be transformed to these two variables. If the Cauchy-Riemann equations which relate the real and imaginary parts of an analytic function are used, it can be shown* that the imaginary parts of the complex functions $U(\zeta)$, $V(\zeta)$, and $W(\zeta)$, and thus the complex functions themselves, also satisfy Eqs. (65) and (66). From this, the first of the vorticity relations can be written

$$\frac{dU}{d\zeta}\frac{\partial \zeta}{\partial y} = \frac{dV}{d\zeta}\frac{\partial \zeta}{\partial x} \tag{67}$$

From Eq. (23),

$$\left.\begin{aligned} \frac{\partial \zeta}{\partial x} &= -\beta \frac{z + iy}{r(x + r)} \\ \frac{\partial \zeta}{\partial y} &= \frac{i\beta}{r(x + r)}\left\{x - \beta^2 \frac{z(z + iy)}{x + r}\right\} \end{aligned}\right\} \tag{68}$$

If Eqs. (68) and (23) are substituted in Eq. (67), it is thus seen that

$$\frac{dU}{d\zeta} = \frac{2i\zeta}{\beta(1 - \zeta^2)}\frac{dV}{d\zeta} \tag{69}$$

Similarly the second vorticity relation can be used to show that

$$\frac{dV}{d\zeta} = i\frac{1 - \zeta^2}{1 + \zeta^2}\frac{dW}{d\zeta} \tag{70}$$

The third vorticity relation gives the same result as eliminating V between Eqs. (69) and (70). These equations are the compatibility equations for conical flow. The compatibility equations can also be written in the more symmetrical form,

$$\left.\begin{aligned} dU/d\zeta &= -(2\zeta/\beta)F(\zeta) \\ dV/d\zeta &= i(1 - \zeta^2)F(\zeta) \\ dW/d\zeta &= (1 + \zeta^2)F(\zeta) \end{aligned}\right\} \tag{71}$$

where the analytic function $F(\zeta)$ is common to all three derivatives.

If a planar wing, near the plane $z = 0$, has straight edges and the boundary conditions have—either in terms of the normal velocity w (the wing shape) or in terms of the streamwise velocity u (the perturbation pressure—a conical distribution with respect to a corner where the wing edges intersect, the flow will be conical in the neighborhood of the corner unless the corner is in the zone of influence of other parts of the wing. If the wing shape is given, the function $W(\zeta)$ can then be determined by conformal mapping. The function $U(\zeta)$ (and thus the lift and drag) can then be determined from Eq. (71). If the perturbation pressures are given, the process is reversed. Most of the important steady-state problems of supersonic wings have been most easily solved by these procedures. A few, such as the lift of a delta wing with subsonic leading edges,[24] can be solved by other methods only with great difficulty. A comprehensive summary of conical flow methods which contains many examples has been prepared by Lagerstrom.[26] A special problem arises in the calculation of drag due to lift if subsonic leading edges are involved since, as in the case of a two-dimensional flat plate wing in an incompressible flow, the solution shows a local singularity corresponding to a finite leading-edge suction which decreases the drag. This

* The analysis is straightforward but lengthy and is not repeated here. For details, see reference 24 where the analysis is used to obtain Eqs. (69) and (70) directly. The given statement then follows directly from this result.

problem has been treated by von Kármán[19] and by Puckett and Stewart.[27] A similar difficulty in the calculation of drag due to the thickness if the leading edges are rounded has been treated by Jones.[28]

Since the velocity potential in any conical flow is a homogeneous function of order 1 in the Cartesian coordinates,

$$\varphi = x(\partial\varphi/\partial x) + y(\partial\varphi/\partial y) + z(\partial\varphi/\partial z) \qquad (72)$$

If the boundary conditions make it possible to determine one of the velocity components by a conformal mapping procedure, then Eqs. (71) and (72) determine the complete flow. These two relations thus replace the formal integration which otherwise would be required. In effect, these two relations make it possible to construct the appropriate function $f_1(\mu, \theta)$ [see Eq. (10)].

The same procedure can be used to construct the appropriate solution when n is any positive integer. As an example, consider the case for $n = 2$. In this case, the condition of homogeneity requires that

$$2\varphi = x(\partial\varphi/\partial x) + y(\partial\varphi/\partial y) + z(\partial\varphi/\partial z) \qquad (73)$$

The first derivatives are homogeneous functions of order 1, and the condition of homogeneity may be applied again. Thus

$$2\varphi = x^2\frac{\partial^2\varphi}{\partial x^2} + y^2\frac{\partial^2\varphi}{\partial y^2} + z^2\frac{\partial^2\varphi}{\partial z^2} + 2xy\frac{\partial^2\varphi}{\partial x\partial y} + 2xz\frac{\partial^2\varphi}{\partial x\partial z} + 2yz\frac{\partial^2\varphi}{\partial y\partial z} \qquad (74)$$

The second derivatives are now homogeneous functions of degree 0 which satisfy the Prandtl-Glauert equation. They can thus [see Eq. (24)] be written as the real part of an analytic function of the complex variable ζ. Let

$$\partial^2\varphi/\partial x_i\partial x_j = R\{F_{ij}(\zeta)\} \qquad (75)$$

Now any first derivative is a homogeneous function of order one (like the conical flow velocity potential); so the three complex functions which determine its three Cartesian derivatives must satisfy the compatibility equations, Eq. (71). Thus, for the derivatives obtained from $\partial\varphi/\partial x$,

$$\left.\begin{aligned} dF_{11}/d\zeta &= -(2\zeta/\beta)F_1(\zeta) \\ dF_{12}/d\zeta &= i(1-\zeta^2)F_1(\zeta) \\ dF_{13}/d\zeta &= (1+\zeta^2)F_1(\zeta) \end{aligned}\right\} \qquad (76)$$

Similarly for the other two sets of derivatives,

$$\left.\begin{aligned} dF_{21}/d\zeta &= -(2\zeta/\beta)\,F_2(\zeta) \\ dF_{22}/d\zeta &= i(1-\zeta^2)F_2(\zeta) \\ dF_{23}/d\zeta &= (1+\zeta^2)F_2(\zeta) \end{aligned}\right\} \qquad (77)$$

and

$$\left.\begin{aligned} dF_{31}/d\zeta &= -(2\zeta/\beta)F_3(\zeta) \\ dF_{32}/d\zeta &= i(1-\zeta^2)F_3(\zeta) \\ dF_{33}/d\zeta &= (1+\zeta^2)F_3(\zeta) \end{aligned}\right\} \qquad (78)$$

The three compatibility functions are not independent since $F_{ij} = F_{ji}$. Thus

$$\left.\begin{aligned} F_2 &= -(i\beta/2\zeta)(1-\zeta^2)F_1 \\ F_3 &= -(\beta/2\zeta)(1+\zeta^2)F_1 \end{aligned}\right\} \qquad (79)$$

The three sets of equations, Eqs. (76), (77), and (78), can therefore be written as a single set,

$$\left.\begin{aligned} dF_{11}/d\zeta &= -(4\zeta^2/\beta^2)F(\zeta) \\ dF_{12}/d\zeta &= (2i\zeta/\beta)(1-\zeta^2)F(\zeta) \\ dF_{13}/d\zeta &= (2\zeta/\beta)(1+\zeta^2)F(\zeta) \\ dF_{22}/d\zeta &= (1-\zeta^2)^2F(\zeta) \\ dF_{23}/d\zeta &= -i(1-\zeta^4)F(\zeta) \\ dF_{33}/d\zeta &= -(1+\zeta^2)^2F(\zeta) \end{aligned}\right\} \qquad (80)$$

where

$$F(\zeta) = (\beta/2\zeta)F_1(\zeta) \qquad (81)$$

is the compatibility function for $n = 2$. The integration procedure then parallels the conical flow case. If one of the complex functions is determined from the boundary conditions, the rest are found from Eq. (80), and the various integrals are given by the condition of homogeneity. The procedure can obviously be applied directly for higher values of n; however, the degree of complexity increases rapidly. For example, if $n = 3$, a family of ten analytic functions must be determined to define the velocity potential.

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